

Date: 25/08/2019

Max. Marks: 102

SOLUTIONS

Time allowed: 3 hours

1. Consider the sequence of numbers $\left[n + \sqrt{2n} + \frac{1}{2} \right]$ for $n \geq 1$, where $[x]$ denotes the greatest integer not exceeding x . If the missing integers in the sequence are $n_1 < n_2 < n_3 < \dots$ then find n_{12} .

Ans. (91)

Sol. $\left[n + \sqrt{2n} + \frac{1}{2} \right] = \left[\left(\sqrt{n} + \frac{1}{\sqrt{2}} \right)^2 \right]$

Let $P = \left[\left(\sqrt{n} + 0.7 \right)^2 \right] \rightarrow \text{GIF}$

- Given ($n \geq 1$), put
- $n = 1 \rightarrow P = 2$
 - $n = 2 \rightarrow P = 4$
 - $n = 3 \rightarrow P = 5$
 - $n = 4 \rightarrow P = 7$
 - $n = 5 \rightarrow P = 8$
 - $n = 6 \rightarrow P = 9$
 - $n = 7 \rightarrow P = 11$

Here we can see that missing number are 3, 6, 10, ...

which is following a certain pattern

Missing Number, $\left(\begin{array}{cccccc} & +3 & +4 & +5 & +6 & +7 \\ 3, & 6, & 10, & 15, & 21, & 28, & 36, & 45, & 55, & 66, & 78, & 91 \end{array} \right)$

Hence $n_{12} = 91$

Note : It should be given that n must be positive integer otherwise the questions is bonus.

2. If $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a + b + c + d|$?

Ans. (93)

Sol. $x = \sqrt{2} + \sqrt{3} + \sqrt{6}$

$x - \sqrt{6} = \sqrt{2} + \sqrt{3}$

$(x - \sqrt{6})^2 = (\sqrt{2} + \sqrt{3})^2$

$x^2 + 6 - 2\sqrt{6}x = 5 + 2\sqrt{6}$

$$x^2 + 6 - 5 = 2\sqrt{6}x + 2\sqrt{6}$$

$$x^2 + 1 = 2\sqrt{6}(x + 1)$$

$$(x^2 + 1)^2 = 24(x^2 + 2x + 1)$$

$$x^4 + 1 + 2x^2 = 24x^2 + 48x + 24$$

$$\Rightarrow x^4 - 22x^2 - 48x - 23 = 0$$

On comparing with equation, $x^4 + ax^3 + bx^2 + cx + d = 0$ we get,

$$a = 0, b = -22, c = -48, d = -23$$

$$\therefore |a + b + c + d| = |0 - 22 - 48 - 23| = 93.$$

3. Find the number of positive integers less than 101 that can not be written as the difference of two squares of integers.

Ans. (25)

Sol. Note that every odd number less than 101 can be written as

$$(k + 1)^2 - (k)^2 = 2k + 1$$

Thus now note that, if any even number can be written as difference of two perfect square, then that number must be a multiple of 4, because

$$2k = a^2 - b^2 = (a + b)(a - b) \equiv 0 \pmod{4} \text{ if } a \text{ and } b \text{ are of same parity.}$$

Also note that, every multiple of 4 can be written as

$$(k + 1)^2 - (k - 1)^2 = 4k$$

Hence total number of numbers less than 101 of the form $a^2 - b^2$ are $50 + 25 = 75$.

Hence the answer is 25.

4. Let $a_1 = 24$ and form the sequence a_n , $n \geq 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are

$$24, 2534, 253534, 25353534, \dots$$

What is the least value of n for which a_n is divisible by 99?

Ans. (88)

Sol. Note that, every term in the sequence is of the form

$$a_n = 2(53) (53) \dots (53)4$$

where the number of 53 is the number is $n - 1$. Also if $99|a_n \Leftrightarrow 9|a_n$ and $11|a_n$. Thus, by divisibility rule of 9 we get

$$0 \equiv a_n \equiv 2 + 8(n - 1) + 4 \pmod{9} \Leftrightarrow n \equiv 7 \pmod{9}$$

Now by divisibility of 11 we get,

$$0 \equiv a_n \equiv (2 + 3(n - 1)) - (5(n - 1) + 4) \pmod{11} \Leftrightarrow n \equiv 0 \pmod{11}$$

But the minimum solution to the congruences $a_n \equiv 7 \pmod{9}$

and $a_n \equiv 0 \pmod{11}$ is 88.

Thus, $n = 88$.

5. Let N be the smallest positive integer such that $N + 2N + 3N + \dots + 9N$ is a number all whose digits are equal. What is the sum of the digits of N ?

Ans. (37)

Sol. $N + 2N + 3N + \dots + 9N$
 $= N(1 + 2 + 3 + \dots + 9)$
 $= N \times \frac{9 \times 10}{2} = 45 \times N$

We have to multiply with '45' to a number such that, the resulting number should have all digits same.

Such $N = 12345679$

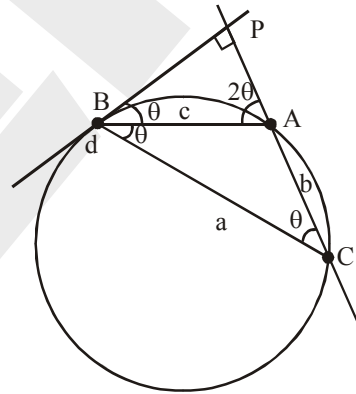
As $45 \times 12345679 = 555555555$

\therefore Sum of digits of $N = 37$

6. Let ABC be a triangle such that $AB = AC$. Suppose the tangent to the circumcircle of ΔABC at B is perpendicular to AC . Find $\angle ABC$ measured in degrees.

Ans. (30)

Sol. Let tangent at B intersect AC at P . Now,
 note that $\angle ABC = \angle ACB = \theta \Rightarrow \angle BAP = 2\theta$
 $\Rightarrow \angle PBA = \theta$,
 In ΔPBA
 $3\theta = 90^\circ$
 $\theta = 30^\circ$



7. Let $s(n)$ denote the sum of the digits of a positive integer n in base 10. If $s(m) = 20$ and $s(33m) = 120$, what is the value of $s(3m)$?

Ans. (60)

Sol. We will take sum of digit base 10 to (mod 9)

Also $s(ab) = s(a) \cdot s(b) \pmod{9}$

Now $s(m) = 20$

$s(33m) = 120 = s(11) \times s(3m)$

$120 = 2 \times s(3m) \quad [\because s(11) = 2 \pmod{9}]$

$60 = s(3m)$

Hence, $s(3m) = 60$

8. Let $F_k(a, b) = (a + b)^k - a^k - b^k$ and let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. For how many ordered pairs (a, b) with $a, b \in S$ and $a \leq b$ is $\frac{F_5(a, b)}{F_3(a, b)}$ an integer?

Ans. (19)

Sol.
$$\frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3}$$

$$\Rightarrow \frac{a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 - a^5 - b^5}{a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3}$$

$$\Rightarrow \frac{5ab[a^3 + 2a^2b + 2ab^2 + b^3]}{3ab[a + b]}$$

$$\Rightarrow \frac{5[a^3 + 3a^2b + 3ab^2 + b^3 - a^2b - ab^2]}{3(a + b)}$$

$$\Rightarrow \frac{5[(a + b)^3 - ab(a + b)]}{3(a + b)}$$

$$\Rightarrow \frac{5[(a + b)\{(a + b)^2 - ab\}]}{3(a + b)}$$

$$\Rightarrow \frac{5}{3}[a^2 + 2ab + b^2 - ab]$$

$$\Rightarrow \frac{5}{3}[a^2 + ab + b^2]$$

Now $3 \mid a^2 + ab + b^2$

Both a and b cannot simultaneously be even.

Three cases are possible

Case 1 : $a \equiv 0 \pmod{3}$

$$b \equiv 0 \pmod{3}$$

$$\therefore (a, b) = (3, 3); (6, 6); (9, 9) \Rightarrow 3 \text{ pairs}$$

Case 2 : $a \equiv 1 \pmod{3}$

$$b \equiv 1 \pmod{3}$$

$$\begin{aligned} \therefore (a, b) = & (1, 1); (1, 4); (1, 7); (1, 10); \\ & (4, 4); (4, 7); (4, 10); \\ & (7, 7); (7, 10); \\ & (10, 10) \end{aligned} \Rightarrow 10 \text{ pairs}$$

Case 3 : $a \equiv 2 \pmod{3}$

$$b \equiv 2 \pmod{3}$$

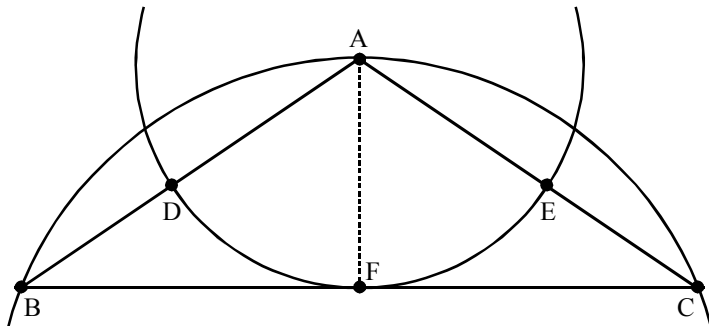
$$\begin{aligned} \therefore (a, b) = & (2, 2); (2, 5); (2, 8) \Rightarrow 6 \text{ pairs} \\ & (5, 5); (5, 8); \\ & (8, 8) \end{aligned}$$

$$\therefore \text{total} = 3 + 10 + 6 = 19 \text{ pairs}$$

9. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α° and the smaller one β° then what is the value of $\alpha - \beta$?

Ans. (90)

Sol.



The center of any circle with D and E on it must pass through the (potentially extended) bisector of $\angle A$. For this center to be on the circumcircle of $\triangle ABC$, the only possibility is for the center to be A itself.

$AF = AD$ since they are both radii of the same circle.

$AD = BD$ since D is the midpoint of \overline{AB} .

$AF \perp BC$, since $\triangle ABC$ is isosceles.

Therefore, since $AB = 2AF$, $\angle B = 30^\circ$. That makes $\angle A = 120^\circ$, so the difference between them is 90° .

10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2 digit number obtained by reversing the digits of x . How many minutes did I walk?

Ans. (42)

Sol. Let $x = ab$

where a and b are the unit digits

$$\begin{aligned} \text{Time after 5'O clock} &= 5 \times 60 + 10a + b \\ &= (300 + 10a + b) \text{ minutes} \end{aligned}$$

$$\begin{aligned} \text{Time after 6'O clock} &= 6 \times 60 + 10b + a \\ &= (360 + 10b + a) \text{ minutes} \end{aligned}$$

$$\therefore 360 + 10b + a - 300 - 10a - b = 10a + b$$

$$60 + 9b - 9a = 10a + b$$

$$60 + 8b = 19a$$

$$\therefore b = \frac{19a - 60}{8}$$

Now a & b are integers from (1 to 9) by putting different values of a , we get $a = 4$, $b = 2$

$\therefore x = 42$ minutes.

11. Find the largest value of a^b such that the positive integers $a, b > 1$ satisfy $a^b b^a + a^b + b^a = 5329$.

Ans. (64)

Sol. $a, b > 1$

$$a^b b^a + a^b + b^a = 5329$$

$$a^b b^a + a^b + b^a + 1 = 5330$$

$$(a^b + 1)(b^a + 1) = 2 \times 5 \times 13 \times 41 \\ = 65 \times 82$$

$$b^a + 1 = 82$$

$$b^a = 81 = 3^4$$

$$b = 3, a = 4$$

$$\text{Now } a^b = 4^3 = 64 \text{ as } a^b + 1 = 65$$

$$a^b = 64$$

\therefore Largest values of $a^b = 64$

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set

$$\{10a + b : 1 \leq a \leq 5, 1 \leq b \leq 5\}$$

where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Ans. (47)

Sol. $10a + b ; 1 \leq a \leq 5, 1 \leq b \leq 5$

Let us divide numbers into different sets, such as

$$\text{Set 1 : } \{11, 12, 13, 14, 15\}$$

$$\text{Set 2 : } \{21, 22, 23, 24, 25\}$$

$$\text{Set 3 : } \{31, 32, 33, 34, 35\}$$

$$\text{Set 4 : } \{41, 42, 43, 44, 45\}$$

$$\text{Set 5 : } \{51, 52, 53, 54, 55\}$$

Now to make a number having no two unit's digit and no two ten's digit same, we can select any 1 number from each of set 1, set 2, set 3, set 4, set 5

$$\therefore \text{ Number of ways : } 5_{C_1} \times 4_{C_1} \times 3_{C_1} \times 2_{C_1} \times 1_{C_1} \\ = 5! = 120$$

$$\therefore 120 \div 73 \Rightarrow \text{Remainder} = 47$$

13. Consider the sequence

$$1, 7, 8, 49, 50, 56, 57, 343, \dots$$

which consists of sums of distinct powers of 7, that is $7^0, 7^1, 7^0 + 7^1, 7^2, \dots$, in increasing order. At what position will 16856 occur in this sequence?

Ans. (34)

Sol.

$$\begin{bmatrix} 7^0 \\ 7^1 \\ 7^0+7^1 \\ 7^2 \\ 7^0+7^2 \\ 7^1+7^2 \\ 7^0+7^1+7^2 \\ 7^3 \\ 7^0+7^3 \\ 7^1+7^3 \end{bmatrix}
 \begin{bmatrix} 7^2+7^3 \\ 7^0+7^1+7^3 \\ 7^0+7^2+7^3 \\ 7^1+7^2+7^3 \\ 7^0+7^1+7^2+7^3 \\ 7^4 \\ 7^0+7^4 \\ 7^1+7^4 \\ 7^2+7^4 \\ 7^3+7^4 \\ 7^0+7^1+7^4 \end{bmatrix}
 \begin{bmatrix} 7^0+7^2+7^4 \\ 7^0+7^1+7^2+7^4 \\ 7^0+7^3+7^4 \\ 7^1+7^2+7^4 \\ 7^1+7^3+7^4 \\ 7^0+7^1+7^3+7^4 \\ 7^0+7^2+7^3+7^4 \\ 7^2+7^3+7^4 \\ 7^1+7^2+7^3+7^4 \\ 7^5 \\ 7^0+7^5 \\ 7^1+7^5 \\ 7^2+7^5 \end{bmatrix} = 16856$$

i.e. we get 16856 at 34th position

14. Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines $x = 4$ and $y = 3$ divide R into four regions R_i , $i = 1, 2, 3, 4$. If $[R_i]$ denotes the area of the region R_i and if $[R_1] > [R_2] > [R_3] > [R_4]$, determine $[R_1] - [R_2] - [R_3] + [R_4]$. (Here $[\Omega]$ denotes the area of the region Ω in the plane.)

Ans. (48)

Sol. We have $\Sigma[R_i] = 36\pi$

$$\begin{aligned}
 [R_1] - [R_2] - [R_3] + [R_4] &= 2([R_1] - [R_2]) - \Sigma[R_i] \\
 &= 2([R_1] + [R_4]) - 36
 \end{aligned}$$

$$\text{Now } \cos \alpha = \frac{2}{3}$$

$$\text{Also, } \theta = \frac{7\pi}{6} - \alpha$$

$$[R_1] = \frac{1}{2} \times 36 \times \left(\frac{7\pi}{6} - \alpha \right) + \frac{1}{2} (4 + 3\sqrt{3}) \times 3 + \frac{1}{2} \times (3 + 2\sqrt{5}) \times 4$$

$$\text{Now } \angle BOP = \alpha \text{ as } \cos \alpha = \frac{2}{3}$$

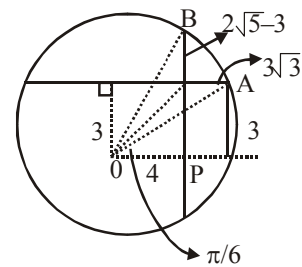
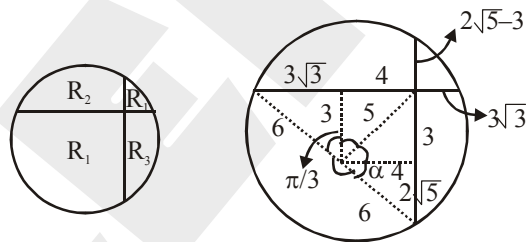
$$\Rightarrow \angle BOA = \alpha - \frac{\pi}{6}$$

$$\text{so, } [R_4] = \frac{1}{2} \times 36 \left(\alpha - \frac{\pi}{6} \right) - \frac{1}{2} \times (2\sqrt{5} - 3) \times 4 - \frac{1}{2} (3\sqrt{3} - 4) \times 3$$

$$\text{Thus } [R_1] + [R_4] = 18\pi + 12 + 12 = 24 + 18\pi$$

Som required answer

$$= 48$$



15. In base -2 notation, digits are 0 and 1 only and the places go up in powers of -2 . For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base -2 how many non zero digits does it contain?

Ans. (06)

Sol. $2019 = 2048 - 32 + 4 - 2 + 1$
 $= (-2)^{12} + (-2)^{11} + (-2)^5 + (-2)^1 + (-2)^0$
 $= 4096 - 2048 - 32 + 4 - 2 + 1$
 $= 1100000100111$ (in base -2)
 Number of non-zero digits = 6

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N ?

Ans. (69)

Sol. n = natural number.

p = prime number

$p < 20$

$\therefore p^2 < 400$

also $\sqrt{n} < p$

$\therefore n < p^2 < 400$

so $n \in \{1, 2, \dots, 399\}$

If $p = 2$, then $n < 2^2 \Rightarrow n < 4$

$\therefore n = 2$ only case $\Rightarrow 1$ solution

If $p = 3$, then $n < 3^2 \Rightarrow n < 9$

$\therefore n = 3, n = 6 \Rightarrow 2$ solution

If $p = 5$, then $n < 5^2 \Rightarrow n < 25$

$\therefore n = 5, 10, 15, 20 \Rightarrow 4$ solution

If $p = 7$, then $n < 7^2 \Rightarrow n < 49$

$\therefore n = 7, 14, 21, 28, 35, 42 \Rightarrow 6$ solution

If $p = 11$, then $n < 11^2 \Rightarrow n < 121$

$\therefore n = 11, 22, 33, 44, 55, 66, 77, 88, 99, 110 \Rightarrow 10$ solution.

If $p = 13$, then $n < 13^2 \Rightarrow n < 169$

$\therefore n = 13, 26, 39, \dots, 156 \Rightarrow 12$ solution.

If $p = 17$, then $n < 17^2 \Rightarrow n < 289$

$\therefore n = 17, 34, \dots, 272 \Rightarrow 16$ solution.

If $p = 19$, then $n < 19^2 \Rightarrow n < 361$

$\therefore n = 19, 38, \dots, 342 \Rightarrow 18$ solution.

Total $1 + 2 + 4 + 6 + 10 + 12 + 16 + 18 \Rightarrow 69$

17. Let a, b, c be distinct positive integers such that $b + c - a, c + a - b$ and $a + b - c$ are all perfect squares. What is the largest possible value of $a + b + c$ smaller than 100?

Ans. (91)

Sol. Let $b + c - a = x^2 \dots$ (i)

$$c + a - b = y^2 \dots$$
 (ii)

$$a + b - c = z^2 \dots$$
 (iii)

Now since a, b, c are distinct positive integers,

$\therefore x, y, z$ will also be positive integers,

add (i), (ii) and (iii)

$$a + b + c = x^2 + y^2 + z^2$$

Now, we need to find largest value of $a + b + c$ or $x^2 + y^2 + z^2$ less than 100.

Now, to get a, b, c all integers x, y, z all must be of same parity, i.e. either all three are even or all three are odd.

Let us maximize $x^2 + y^2 + z^2$, for both cases.

If x, y, z are all even.

$$\Rightarrow b + c - a = 8^2 = 64$$

$$c + a - b = 4^2 = 16$$

$$a + b - c = 2^2 = 4$$

Which on solving, give $a = 10, b = 34, c = 40$ and $a + b + c = 84$

If x, y, z are all odd

$$\Rightarrow b + c - a = 9^2 = 81$$

$$c + a - b = 3^2 = 9$$

$$a + b - c = 1^2 = 1$$

Which on solving, give $a = 5, b = 41, c = 45$ and $a + b + c = 91$

\therefore Maximum value of $a + b + c < 100 = 91$

18. What is the smallest prime number p such that $p^3 + 4p^2 + 4p$ has exactly 30 positive divisors?

Ans. (43)

Sol. $p^3 + 4p^2 + 4p$

$$\Rightarrow p(p^2 + 4p + 4)$$

$$\Rightarrow p(p + 2)^2$$

This number has 30 divisors so it can be in the form of

$$a \cdot b^{14}$$

$$a^2 \cdot b^9$$

$$a \cdot b^2 \cdot c^4$$

$$a^{29}$$

Out of these cases, we will check cases in which p can be minimum which seems to be possible with $a \cdot b^2 \cdot c^4$ case

so by simply checking several cases,

we can put $p = 43$

$$\therefore \Rightarrow 43 (45)^2$$

$$\Rightarrow 43 \times 15^2 \times 3^2$$

$$\Rightarrow 43 \times 5^2 \times 3^2 + 3^2$$

$$\Rightarrow 43 \times 5^2 \times 3^4$$

Whose no. of divisors are

$$(1 + 1) \cdot (2 + 1) \cdot (4 + 1)$$

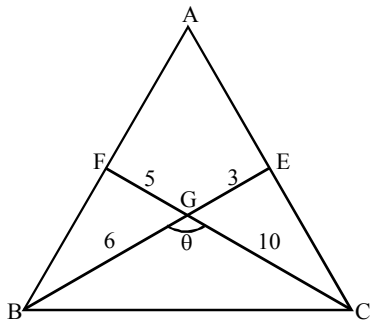
$$2 \times 3 \times 5 = 30$$

$$\therefore p = 43$$

- 19.** If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

Ans. (90)

Sol.



$$\text{Area of } \triangle BGC = \frac{1}{2} \times 6 \times 10 \times \sin\theta$$

To maximize the area of $\triangle BGC$, $\sin\theta = 1$

$$\therefore \text{maximum area of } \triangle BGC = 30$$

$$\text{Maximum area of } \triangle ABC = 3 \triangle BGC$$

$$= 3 \times 30 = 90 \text{ sq. units}$$

- 20.** How many 4-digit numbers \overline{abcd} are there such that $a < b < c < d$ and $b - a < c - b < d - c$?

Ans. (07)

Sol. \overline{abcd}

$$\therefore a < b < c < d$$

$$\therefore a \geq 1$$

$$b \geq 2$$

$$c \geq 3$$

$$d \geq 4$$

$$\text{Also } b - a < c - b$$

$$\text{i.e. } 2b < a + c$$

$$\text{and } c - b < d - c$$

$$2c < b + d$$

We can make a table applying all these conditions

a	b	c	d
1	2	4	7
1	2	4	8
1	2	4	9
1	2	5	9
2	3	5	8
2	3	5	9
3	4	6	9

So total 7 numbers are possible.

21. Consider the set E of all positive integers n such that when divided by 9, 10, 11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

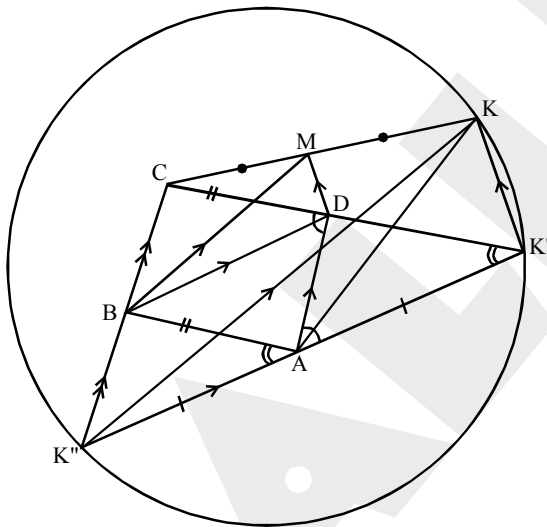
Ans. (Bonus)

Sol. Question is incorrect.

22. In parallelogram ABCD, $AC = 10$ and $BD = 28$. The points K and L in the plane of ABCD move in such a way that $AK = BD$ and $BL = AC$. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

Ans. (02)

Sol.



Produce CD to K' such that $CD = DK'$

then $BDK'A$ is a parallelogram

$$\therefore AB = CD = DK'$$

$$AB \parallel DK'$$

$$\therefore AK' = BD$$

Draw a circle with centre A and radius BD which cuts CD produced at K' and CB produced at K''

then $K''AK'$ are collinear as $\angle CDA + \angle BDA = 180^\circ$

$$\angle CDA = \angle DAK' + \angle DK'A = \angle DAK' + \angle BAK'' \quad [\because BA \parallel DK']$$

$$\therefore \angle DAK' + \angle BAK'' + \angle BAD = 180^\circ$$

Thus $K'AK''$ is a diameter.

Let K is any point on this circle

Since M is a mid point of CK

D is a mid point of CK'

then $MD \parallel KK'$

In $\triangle CK'K''$,

D is a mid point of CK'

$DB \parallel K'A$ i.e, $DB \parallel K'K''$

$\therefore B$ is a mid point of CK''

In $\triangle CK''K$

B, M are the mid points of CK'' and CK respectively.

\therefore In $\triangle BMD$ and $\triangle K''KK'$

$BM \parallel K''K$

$MD \parallel KK'$

$BD \parallel K''K'$

$\therefore \angle BMD = \angle K''KK' = 90^\circ$

$\therefore K''K'$ is a diameter similarly for other \triangle

$\angle ANC = 90^\circ$

So $\frac{\angle BMD}{2} = 45^\circ, \frac{\angle ANC}{2} = 45^\circ$

$$\cot^2\left(\frac{\angle BMD}{2}\right) + \tan^2\left(\frac{\angle ANC}{2}\right) = 1 + 1 = 2$$

23. Let t be the area of a regular pentagon with each side equal to 1. Let $P(x) = 0$ be the polynomial equation with least degree, having integer coefficients, satisfied by $x = t$ and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of $P(x)$. What is the integer closest to \sqrt{M} ?
 ($\sin 18^\circ = (\sqrt{5}-1)/2$).

Ans. (Bonus)

Sol. Area of regular polygon = $\frac{a^2 n}{4 \tan\left(\frac{180}{n}\right)}$

n = number of sides

a = length of side

\therefore for regular pentagon of side length 1,

$$\text{area} = t = \frac{5}{4 \tan 36^\circ} = \frac{5}{4(0.73)} \cong 1.71$$

Now, $P(1.71) = 0$ to be found with least degree and integer coefficient such that gcd of all coefficients is 1.

$$\text{Let } x = 1.71$$

$$100x = 171$$

$\therefore P(x) = 100x - 171 = 0$ is the required polynomial

Which satisfied all the conditions.

$$\therefore m = |100| + |-171| = 271$$

$$\therefore \sqrt{m} = 16.46$$

$$\therefore \text{nearest integer} = 16$$

But this question can have multiple solutions as student can take $\tan 36^\circ$ as 0.72, 0.726 or even 0.7, every time we will get different answers. So this question should be Bonus.

24. For $n \geq 1$, let a_n be the number beginning with n 9's followed by 744; e.g., $a_4 = 9999744$.

Define $f(n) = \max \{m \in \mathbb{N} \mid 2^m \text{ divides } a_n\}$, for $n \geq 1$. Find $f(1) + f(2) + f(3) + \dots + f(10)$.

Ans. (75)

Sol. $a_1 = 9744$

$$a_2 = 99744$$

$$a_3 = 999744$$

and so on...

$$\therefore 9744 \text{ is divisible by } 16$$

$$\therefore \text{Each } a_n \text{ is divisible by at least } 2^4.$$

$$\text{Now, } a_1 = 10^4 - 256 \equiv 0 \pmod{2^4}$$

$$a_2 = 10^5 - 256 \equiv 0 \pmod{2^5}$$

$$a_3 = 10^6 - 256 \equiv 0 \pmod{2^6}$$

$$a_4 = 10^7 - 256 \equiv 0 \pmod{2^7}$$

$$a_5 = 10^8 - 256 = 256(390625 - 1)$$

$$= 256 \times 390264$$

$$= 256 \times 32 \times 12207$$

$$= 2^{13} \times 12207$$

$$\equiv 0 \pmod{2^{13}}$$

$$a_6 = 10^9 - 256 \equiv 0 \pmod{2^8}$$

$$a_7 = 10^{10} - 256 \equiv 0 \pmod{2^8}$$

$$a_8 = 10^{11} - 256 \equiv 0 \pmod{2^8}$$

$$a_9 = 10^{12} - 256 \equiv 0 \pmod{2^8}$$

$$a_{10} = 10^{13} - 256 \equiv 0 \pmod{2^8}$$

$$\therefore f(1) + f(2) + \dots + f(10)$$

$$= 4 + 5 + 6 + 7 + 13 + 8 + 8 + 8 + 8 + 8$$

$$= 75$$

25. Let ABC be an isosceles triangle with AB = BC. A trisector of $\angle B$ meets AC at D. If AB, AC and BD are integers and $AB - BD = 3$, find AC.

Ans. (26)

Sol. Let $B = 6\theta$

Let $BD = x \in \mathbb{Z}$

$\Rightarrow AB = x + 3$

Given $AC \in \mathbb{Z}$

$A = (3 + x) \cos 3\theta = x \cos\theta$

$\Rightarrow (3 + x)(4\cos^2\theta - 3) = x$

$\Rightarrow \sin^2\theta = \frac{3}{4(x+3)}$

Now $AL = (3 + x) \sin\theta$

$\Rightarrow AC = 2(3 + x)(3\sin\theta - 4\sin^3\theta)$

$$= 2(3 + x) \sin\theta \left(3 - 4 \frac{3}{4(x+3)}\right)$$

$$= 6(2 + x) \frac{\sqrt{3}}{2\sqrt{x+3}} = 3(x + 2) \frac{\sqrt{3}}{\sqrt{x+3}}$$

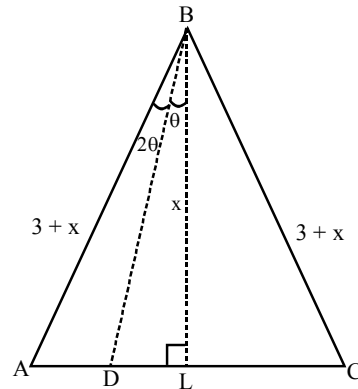
$\Rightarrow x = 3y \quad \Rightarrow AC = \frac{3(3y+2)}{\sqrt{y+1}}$

$\Rightarrow y + 1 = z^2 \quad \Rightarrow AC = \frac{3(3z^2 - 1)}{z} = 9z - \frac{3}{z}$

$\Rightarrow z = 1$ or 3

But $z \neq 1$ as $x = 0$ not possible

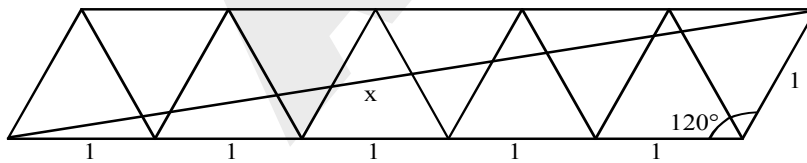
$\Rightarrow z = 3 \quad \Rightarrow AC = 26$



26. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance travelled by the ball in meters is of the form \sqrt{N} , where N is an integer. What is the value of N?

Ans. (31)

Sol.



$$x^2 = 5^2 + 1^2 - 2 \times 5 \times 1 \cos 120^\circ$$

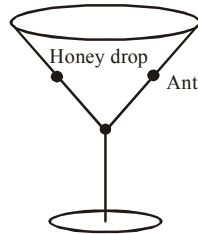
$$= 25 + 1 + 5$$

$$x = \sqrt{31}$$

$$N = 31$$

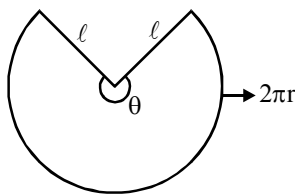
Folding the triangle continuously each time of reflection creates the above diagram. 9 points of reflection can be verified in the diagram above. Thus root (N) is the length of red line which is root (31). Thus $N = 31$ is the answer.

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass (See the figure.). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d ? (Ignore the thickness of the glass.)

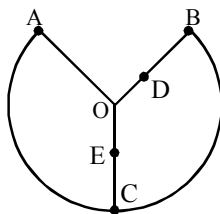


Ans. (38)

Sol. Let's cut the cone along slant height of cone.



$$\text{Now, } \ell\theta = 2\pi r \quad \theta = \frac{4\pi}{3}$$



When A goes to B (fixed) the major arc becomes base circle of the cone i.e. may \widehat{AB} becomes full circle. So, mid-point of major \widehat{AB} i.e. point C will be diametrically opposite end of fixed point B.

Now, Ant lies at the mid-point of OB (above the plane) i.e. point D and she should reach point E (below the plane). So, required shortest path is D to C then C to E.

$$\text{(Note } \angle DOC = \frac{2\pi}{3}\text{)}$$

$$\text{So, } d = DC + CE$$

$$= \sqrt{\ell^2 + \frac{\ell^2}{4} + \frac{\ell^2}{2} + \frac{\ell}{2}} + \frac{\ell}{2} = \frac{\sqrt{7}\ell}{2} + \frac{\ell}{2} = \frac{(\sqrt{7}+1)21}{2}$$

$$\therefore [d] = 38$$

28. In a triangle ABC, it is known that $\angle A = 100^\circ$ and $AB = AC$. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10^\circ \approx 0.174$.

Ans. (27)

Sol. Given, $BD = 20$ units

$$\angle A = 100^\circ$$

$$AB = AC$$

In $\triangle ABD$

$$\frac{BD}{\sin A} = \frac{AD}{\sin 2\theta}$$

$$\frac{BD}{\sin 100^\circ} = \frac{AD}{\sin 20^\circ}$$

$$\frac{BD}{\cos 10^\circ} = \frac{AD}{2 \sin 10^\circ \cos 20^\circ}$$

$$\Rightarrow 20 = \frac{AD}{2 \sin 10^\circ} \Rightarrow AD = 40 \cdot \sin 10^\circ = 6.96$$

In $\triangle BDC$

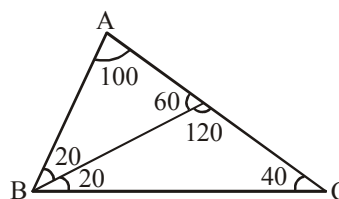
$$\text{Also, } \frac{BD}{\sin 40^\circ} = \frac{BC}{\sin 120^\circ} = \frac{CD}{\sin 20^\circ}$$

$$\frac{20}{2 \sin 20^\circ \cdot \cos 20^\circ} = \frac{CD}{\sin 20^\circ} \Rightarrow CD = \frac{20}{2 \cos 20^\circ} = \frac{20}{2 \times 0.9394} \cong 10.65$$

$$\therefore AD + CD = AC = AB \cong 17.6$$

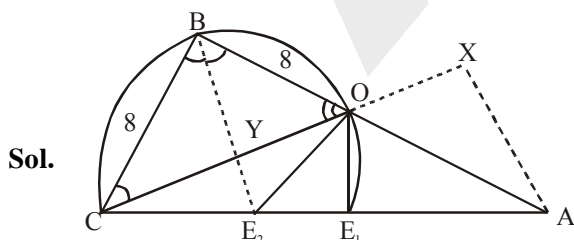
Now since BD is angle bisector

$$\text{So } \frac{BC}{AB} = \frac{CD}{AD} \Rightarrow BC = \frac{AB \times CD}{AD} = \frac{17.6 \times 10.65}{6.96} \cong 26.98 \cong 27$$



29. Let ABC be an acute angled triangle with $AB = 15$ and $BC = 8$. Let D be a point on AB such that $BD = BC$. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible values of AE, find $[\alpha]$ the integer part of α .

Ans. (68)



The pairs E_1, E_2 satisfies condition or $E_1 =$ intersection of CBO with AC and $E_2 =$ intersection of \angle bisector of B and AC

\therefore that $\angle DE_2B = \angle CE_2B$ and for E_1 $\angle BE_1C = \angle BDC = \angle BCD = \angle BE_1D$
 $\Rightarrow \overline{AE_1} \cdot \overline{AC} = \overline{AD} \cdot \overline{AB} = 7 \times 15$

$$\frac{\overline{AE_2}}{\overline{AC}} = \frac{XY}{XC}$$

(For y is midpoint of OC and X is foot of altitude from A to CD)

Also $\frac{XD}{DY} = \frac{7}{8}$ and $DY = YC$

$$\Rightarrow \frac{XD+DY}{XC} = \frac{15}{7+8+8} = \frac{15}{23}$$

$$\Rightarrow \frac{XY}{XC} = \frac{15}{23} \Rightarrow \frac{AE_2}{AC} = \frac{15}{23}$$

$$\Rightarrow AE_1 \cdot AE_2 = \frac{15}{23} \cdot 7 \cdot 15 = \frac{225 \times 7}{23}$$

Ans. $\left[\frac{225 \times 7}{23} \right] = 68$

- 30.** For any real number x , let $[x]$ denote the integer part of x ; $\{x\}$ be the fractional part of x . ($\{x\} = x - [x]$). Let A denote the set of all real numbers x satisfying

$$\{x\} = \frac{x + [x] + [x + (1/2)]}{20}$$

If S is the sum of all numbers in A , find $[S]$.

Ans. (21)

Sol. $\{x\} = \frac{x + [x] + \left[x + \frac{1}{2} \right]}{20} \Rightarrow 20f = 2I + f + \left[x + \frac{1}{2} \right]$

$$\Rightarrow 19f = 2I + \left[x + \frac{1}{2} \right]$$

Let $x = I + F = [x] + \{x\}$

case I: $0 \leq f < \frac{1}{2}$ i.e. $\left[x + \frac{1}{2} \right] = I$

so, $19f = 3I \in \left[0, \frac{19}{2} \right)$

$$\Rightarrow I \in \left[0, \frac{19}{6} \right)$$

Hence, $x = I + f = I + \frac{3I}{19} = \frac{22I}{19}$;

$$I = 0, 1, 2, 3$$

$$\text{Case II: } f \in \left[\frac{1}{2}, 1\right) \quad \text{i.e. } \left[x + \frac{1}{2}\right] = I + 1$$

$$\text{so, } 19f = 3I + 1 \in \left[\frac{19}{2}, 19\right)$$

$$\Rightarrow I \in \left[\frac{17}{6}, 6\right)$$

$$\Rightarrow I = 3, 4, 5$$

$$\text{Hence, } x = I + f = I + \frac{3I+1}{19}$$

$$= \frac{22I}{19} + \frac{1}{19}; I = 3, 4, 5$$

$$\text{Thus, } S = \frac{22}{19} \times 18 + \frac{3}{19} = 21$$

$$[S] = 21$$
