CAREER POINT

JEE Advanced Exam 2019
(Paper & Solution)

Date : 27 / 05 / 2019

PAPER-2

PART-I (MATHS)

SECTION – 1 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct Answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	: **+1**	If two or more options are correct but ONLY one option is chosen and it is a correct.
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 marks;
  choosing ONLY (B) will get +1 marks;
  choosing ONLY (D) will get +1 marks;
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get –1 mark

Q.1 For $a \in R$, $|a| > 1$, let

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+\ldots+\frac{1}{(an+n)^2}\right)}\right)=54.$$

Then the possible value(s) of a is/are

(1) 8 (2) –9 (3) –6 (4) 7

Ans. **[1,2]**

Sol. $$\lim_{n\to\infty}\frac{\left(\left(\frac{1}{n}\right)^{1/3}+\ldots\left(\frac{n}{n}\right)^{1/3}\right)}{\left(\frac{1}{\left(a+\frac{1}{n}\right)^2}+\frac{1}{\left(a+\frac{2}{n}\right)^2}+\ldots+\frac{1}{\left(a+\frac{n}{n}\right)^2}\right)}$$

$$\Rightarrow \frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(a+\frac{r}{n}\right)^2}} \Rightarrow \frac{\int_0^1 (x)^{1/3}dx}{\int_0^1 \frac{1}{(a+x)^2}dx} \Rightarrow \frac{\left(\frac{x^{4/3}}{4/3}\right)_0^1}{\left(-\frac{1}{(a+x)}\right)_0^1} \Rightarrow \frac{\frac{3}{4}}{\left(\frac{1}{a}-\frac{1}{a+1}\right)} = 54$$

$$\Rightarrow \frac{1}{72} = \frac{1}{a(a+1)}$$

$a^2 + a = 72$

$(a + 9)(a - 8) = 0$

$a = 8$ & $a = -9$

Q.2 Let $f(x) = \frac{\sin \pi x}{x^2}$, $x > 0$.

Let $x_1 < x_2 < x_3 < \ldots x_n < \ldots$ be all the points of local maximum of f and

$y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f.

The which of the following options is/are correct?

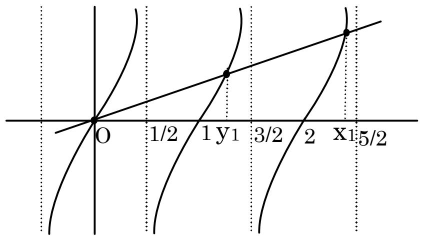
(1) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n (2) $x_1 < y_1$

(3) $|x_n - y_n| > 1$ for every n (4) $x_{n+1} - x_n > 2$ for every n

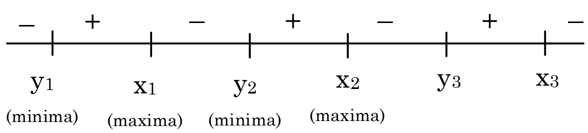
Ans. **[1,3,4]**

Sol. $y = \frac{\sin \pi x}{x^2}$ $\quad x > 0$

$$f'(x) = \frac{2x\cos\pi x\left(\frac{\pi x}{2} - \tan\pi x\right)}{x^4}$$



f ′(x) →



Q.3 Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

and $X = \sum_{k=1}^{6} P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$

where P_k^T denotes the transpose of the matrix P_k. Then which of the following options is/are correct?

(1) X is a symmetric matrix

(2) If $X\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$

(3) The sum of diagonal entries of X is 18

(4) $X - 30I$ is an invertible matrix

Ans. **[1,2,3]**

Sol. $\because P_1 = P_1^T = P_1^{-1}$

$\& \ P_2 = P_2^T = P_2^{-1}$

$\vdots$

$P_6 = P_6^T = P_6^{-1}$

also $A^T = A$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$\because\ X = \sum_{k=1}^{6}(P_k A\ P_k^T)$

$X = (P_1A\ P_1^T + ... + P_6AP_6^T)$

Now $X^T = P_1A^TP_1^T + ... + P_6A^TP_6^T = X \Rightarrow X$ is symmetric

Let $B = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$XB = P_1AP_1^TB + P_2AP_2^TB + ... + P_6AP_6^TB \qquad \because\ P_1^TB = B$

$= P_1AB + ... + P_6AB$

$= (P_1 + P_2 + ... + P_6)AB$

$= 30B \Rightarrow \alpha = 30$

trace of $X = tr(X) = tr(P_1AP_1^T) + tr(P_2AP_2^T) + ... = 3 + 3 + ...3 \Rightarrow 18$

$\because\ X\begin{bmatrix}1\\1\\1\end{bmatrix} = 30\begin{bmatrix}1\\1\\1\end{bmatrix}$

$(X - 30)B = 0$

$|(X - 30)B| = 0$

$|X - 30I|\ |B| = 0$

$|X - 30I| = 0, \quad |B| \neq 0$

Q.4 Let $x \in R$ and let

$P = \begin{bmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 3\end{bmatrix}, Q = \begin{bmatrix}2 & x & x\\0 & 4 & 0\\x & x & 6\end{bmatrix}$ and $R = PQP^{-1}$. Then which of the following options is/are correct?

(1) For $x = 0$, if $R\begin{bmatrix}1\\a\\b\end{bmatrix} = 6\begin{bmatrix}1\\a\\b\end{bmatrix}$, then $a + b = 5$

(2) For $x = 1$, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(3) There exists a real number x such that PQ = QP

(4) $\det R = \det\begin{bmatrix}2 & x & x\\0 & 4 & 0\\x & x & 5\end{bmatrix} + 8$, for all $x \in R$

Ans. **[1,4]**

Sol. $R = PQP^{-1}$

$|R| = |P|\ |Q|\ |P^{-1}|$

$= |Q|$

$= 48 - 4x^2$

Now, $R\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow \alpha = \beta = \gamma = 0$

at x = 0

$R = PQP^{-1}$

$= \begin{bmatrix}2 & 1 & 2/3\\ 0 & 4 & 4/3\\ 0 & 0 & 6\end{bmatrix}$

Now, $(R - 6I)\begin{bmatrix}1\\a\\b\end{bmatrix} = \begin{bmatrix}-4 & 1 & 2/3\\ 0 & -2 & 4/3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\a\\b\end{bmatrix}$

$-4 + a + \frac{2b}{3} = 0$

$-2a + \frac{4b}{3} = 0$

$\Rightarrow a = 2, b = 3$

Now, PQ = QP

$x + 4 + x = 2 + 2x + 0$

$\Rightarrow x \in \phi$

No value

Q.5 Let $f : R \to R$ be a function. We say that f has

PROPERTY 1 if $\lim_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim_{h\to 0}\frac{f(h)-f(0)}{h^2}$ exists and is finite.

Then which of the following options is/are correct?

(1) $f(x) = x^{2/3}$ has PROPERTY1

(2) $f(x) = x|x|$ has PROPERTY 2

(3) $f(x) = \sin x$ has PROPERTY 2

(4) $f(x) = |x|$ has PROPERTY 1

Ans. **[1,4]**

Sol. (1) $y = x^{2/3}$

$\lim_{h\to 0}\frac{h^{2/3}-0}{\sqrt{|h|}} = 0 \to$ finite

(2) $y = x|x|$

$\lim_{h\to 0}\frac{h|h|-0}{h^2} = \lim_{h\to 0}\frac{|h|}{h}$

$$\left.\begin{aligned}&\textbf{Case –I :}\ \lim_{h\to 0^+}\frac{|h|}{h}=1\\&\textbf{Case –II :}\ \lim_{h\to 0^-}\frac{|h|}{h}=-1\end{aligned}\right\}\to \text{limit DNE}$$

(3) $y = \sin x$

$$\lim_{h\to 0}\frac{\sin h-0}{h^2}\to \text{DNE}$$

(4) $y = |x|$

$$\lim_{h\to 0}\frac{|h|-0}{\sqrt{|h|}}=0\to \text{finite}$$

Option (1, 4)

Q.6 Three lines

$L_1 : \vec{r}=\lambda\hat{i}$, $\lambda \in R$,

$L_2 : \vec{r}=\hat{k}+\mu\hat{j}$, $\mu \in R$ and

$L_3 : \vec{r}=\hat{i}+\hat{j}+v\hat{k}$, $v \in R$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear ?

(1) $\hat{k}-\frac{1}{2}\hat{j}$ (2) $\hat{k}+\frac{1}{2}\hat{j}$ (3) $\hat{k}$ (4) $\hat{k}+\hat{j}$

Ans. **[1,4]**

Sol. $P(\lambda, 0, 0)$

$Q(0, \mu, 1)$

$R(1, 1, v)$

$\overrightarrow{PQ} = K\overrightarrow{PR}$

$$\Rightarrow \frac{\lambda}{\lambda-1}=\frac{-\mu}{-1}=\frac{-1}{v}$$

$$\Rightarrow 1+\frac{1}{\lambda-1}=\mu=\frac{1}{v}$$

$\Rightarrow \mu$ can not have values 0 & 1

Q.7 Let $f : R \to R$ be given by $f(x) = (x – 1) (x – 2) (x – 5)$. Define $F(x) = \int_0^x f(t)\,dt$, $x > 0$.

Then which of the following options is/are correct?

(1) F has a local minimum at x = 1

(2) F has two local maxima and one local minimum in $(0, \infty)$

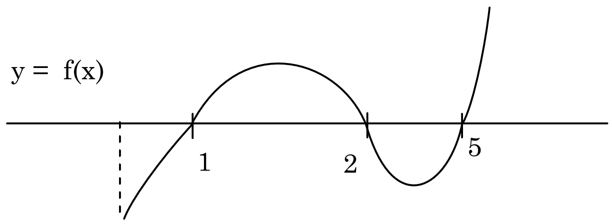
(3) F has a local maximum at x = 2

(4) $F(x) \neq 0$ for all $x \in (0, 5)$

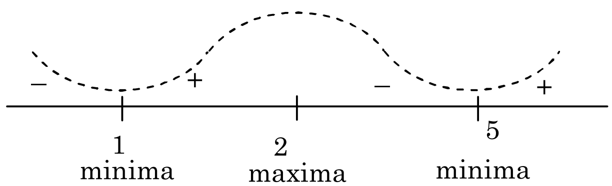
Ans. **[1,3,4]**

Sol. $y = (x-1)(x-2)(x-5)$

$$F(x) = \int_0^x f(t)\,dt$$



$$F'(x) = f(x) = (x-1)(x-2)(x-5)$$



$$f(x) = x^3 - 8x^2 + 17x - 10$$

$$F(x) = \int_0^x f(x)\,dx$$

$F(x) < 0 \ \forall\ x \in (0, 5)$

Q.8 For non-negative integers n, let

$$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}x$ takes values in $[0, \pi]$ which of the following options is/are correct?

(1) $f(4) = \frac{\sqrt{3}}{2}$

(2) $\sin(7\cos^{-1} f(5)) = 0$

(3) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$

(4) $\lim_{n\to\infty} f(n) = \frac{1}{2}$

Ans. **[1,2,3]**

Sol.
$$f(x) = \frac{\sum_{k=0}^{n}\left(\cos\left(\frac{\pi}{n+2}\right) - \cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{\sum_{k=0}^{n}\left(1 - \cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$\Rightarrow \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \frac{\cos\left(\frac{n+3}{n+2}\right)\pi \,.\, \sin\left(\frac{n+1}{n+2}\right)\pi}{\sin\left(\frac{\pi}{n+2}\right)}}{(n+1) - \frac{\cos\pi . \sin\left(\frac{n+1}{n+2}\right)\pi}{\sin\left(\frac{\pi}{n+2}\right)}}$$

$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{(n+1)+1} \Rightarrow \cos\left(\frac{\pi}{n+2}\right)$$

$$f(x) = \cos\left(\frac{\pi}{n+2}\right)$$

(1) $f(4) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(2) $\sin(7\cos^{-1} f(5)) = 0$

(3) $\alpha = \tan\frac{\pi}{8} \Rightarrow \alpha^2 + 2\alpha - 1 = 0$

(4) $\lim\limits_{n\to\infty} \cos\left(\frac{\pi}{n+2}\right) = 1$

SECTION – 2 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

  Full Marks : +3 If ONLY the correct numerical value is entered.

  Zero Marks : 0 In all other cases.

Q.1 Suppose

$$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$$

holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals......

Ans. **[6.20]**

Sol. $\begin{vmatrix} \frac{n(n+1)}{2} & n(n-1)2^{n-2}+n.2^{n-1} \\ n.2^{n-1} & 4^n \end{vmatrix} = 0$

$= \frac{n(n+1)}{2} - \frac{n^2(n-1)}{8} - \frac{n^2}{4} = 0$

$n = 0$ or $4(n + 1) - n(n - 1) - 2n = 0$

$n = 4$

$\therefore \sum_{k=0}^{4} \frac{{}^4C_k}{k+1} = \sum_{k=0}^{4} \frac{{}^5C_{k+1}}{5}$

$\Rightarrow \frac{2^5-1}{5} = \frac{31}{5} = 6.20$

Q.2 Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, α, $\beta \in$ R. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $(\vec{c} - (\vec{a} \times \vec{b})).\vec{c}$ equals .....

Ans. **[18]**

Sol. $\vec{c} = \alpha(2\hat{i} + \hat{j} - \hat{k}) + \beta(\hat{i} + 2\hat{j} + \hat{k})$

$\vec{c} = (2\alpha + \beta)\hat{i} + (\alpha + 2\beta)\hat{j} + (\beta - \alpha)\hat{k}$

$\frac{\vec{c}.(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|} = 3\sqrt{2}$

$9(\alpha + \beta) = 18$

$\alpha + \beta = 2$

$(\vec{c} - \vec{a} \times \vec{b}).\vec{c} = 6(\alpha^2 - 2\alpha + 4)$

Minimum value = 18

Q.3 The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\cos\theta} + \sqrt{\sin\theta})^5} d\theta$ equals ....

Ans. **[0.5]**

Sol. $I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\sin\theta} + \sqrt{\cos\theta})^5} d\theta$ … (1)

$I = \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{(\sqrt{\sin\theta} + \sqrt{\cos\theta})^5} d\theta$ … (2)

(1) + (2)

$$2I = 3\int_0^{\pi/2} \frac{d\theta}{(\sqrt{\sin\theta} + \sqrt{\cos\theta})^4}$$

$$\frac{2I}{3} = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{(\sqrt{\tan\theta}+1)^4}$$

Put $\tan\theta = t^2 \Rightarrow \sec^2\theta\, d\theta = 2t\, dt$

$$\frac{2I}{3} = \int_0^{\infty} \frac{2t\, dt}{(t+1)^4}$$

$$\Rightarrow \frac{I}{3} = \int_0^{\infty}\left[\frac{1}{(t+1)^3} - \frac{1}{(t+1)^4}\right]dt$$

$$I = \frac{1}{2} = 0.5$$

Q.4 Five persons A, B, C, D and E are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green, then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is ....

Ans. **[30]**

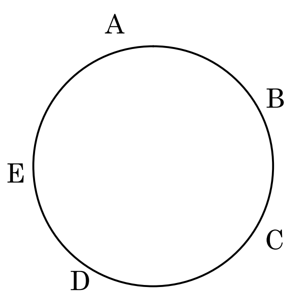
Sol. Combinations can only be

(i) RRGGB

(ii) RGGBB

(iii) RRGBB

3 possibility



No. of ways = $\underset{\text{No. of possibility}}{3} \times \underset{\text{to distribute non repeating hats}}{5} \times \underset{\text{distributing repeating hats}}{2} = 30$

Q.5 The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals ....

Ans. **[0]**

Sol. $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$

$$= \sec^{-1}\left(-\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\text{cosec}\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)$$

$$= \sec^{-1}\left[-\frac{1}{4}\sum_{k=0}^{10}\frac{2}{\sin\left(\frac{7\pi}{6}+k\pi\right)}\right]$$

$$= \sec^{-1}\left[-\frac{1}{2}\sum_{k=0}^{10}\frac{1}{(-1)^{k+1}\sin\frac{\pi}{6}}\right]$$

$$= \sec^{-1}(1) = 0$$

Q.6 Let |X| denote the number of elements in a set X. Let S = {1, 2, 3, 4, 5, 6} be a sample space, where each elements is equally likely to occur. If A and B are independent events associated with S, then the number of ordered pairs (A, B) such that $1 \le |B| < |A|$, equals ....

Ans. **[1523]**

Sol. $1 \le |B| < |A|$

$^6C_1(^6C_2 + {^6C_3} + {^6C_4} + {^6C_5} + {^6C_6}) + {^6C_2}(^6C_3 + ... + {^6C_6}) + {^6C_3}\ (^6C_4 + ...\ {^6C_6}) + {^6C_4}\ (^6C_5 + {^6C_6}) + {^6C_5}(^6C_6)$

$= 6(15 + 20 + 15 + 6 + 1) + 15(20 + 15 + 6 +1) + 20(15 + 6 + 1) + 15(6 + 1) + 6(1)$

$=1523$

SECTION – 3 (Maximum Marks : 12)

- This section contains **Two (02)** List-Match sets.
- Each List-Match set has **TWO (02)** Multiple Choice Questions.
- Each List-Match set has two lists : **List-I** and **List-II**
- **List-I** has **Four** entries (I), (II), (III) and (IV) and **List-II** has **Six** entries (P), (Q), (R), (S), (T) and (U).
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme :

  Full Marks : +3 If ONLY the option corresponding to the correct combination is chosen.

  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered.)

  Negative Marks : –1 In all other cases.

Answer the following by appropriately matching the lists based on the information given in the paragraph.

Q.1 Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$X = \{x : f(x) = 0\}$, $Y = \{x : f'(x) = 0\}$

$Z = \{x : g(x) = 0\}$, $W = \{x : g'(x) = 0\}$

List-I contains the sets X, Y, Z and W. List-II contains some information regarding these sets.

LIST-I	**LIST-II**
(I) X	**(P)** $\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II) Y	**(Q)** an arithmetic progression
(III) Z	**(R)** NOT an arithmetic progression
(IV) W	**(S)** $\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
	(T) $\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
	(U) $\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination?

(1) (II), (Q), (T) (2) (I), (P), (R) (3) (II), (R), (S) (4) (I), (Q), (U)

Ans. **[1]**

Sol. $X = \{x : f(x) = 0\}$

$f(x) = 0 \Rightarrow \sin(\pi \cos x) = 0$

$\Rightarrow \pi \cos x = n\pi, n \in I$

$\Rightarrow \cos x = n$

So $\cos x = -1, 0, 1$

$\Rightarrow X = n\pi, (2n + 1)\frac{\pi}{2}$

OR $X = \left\{n.\frac{\pi}{2}, n \in I\right\}$

$Y = \{x : f'(x) = 0\}$

Now, $f'(x) = 0 \Rightarrow \cos(\pi \cos x)(-\pi \sin x) = 0$

$\Rightarrow \cos(\pi \cos x) = 0$ or $(-\pi \sin x) = 0$

$\Rightarrow \pi \cos x = (2n + 1)\frac{\pi}{2}$ or $x = n\pi$

$\Rightarrow \cos x = (2n + 1)\frac{1}{2}$ or $x = n\pi$

$\Rightarrow \cos x = \pm\frac{1}{2}$ or $x = n\pi$

$Y = \left\{2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \frac{2\pi}{3}, n\pi, n \in I\right\}$

$Z = \{x : g(x) = 0\}$

$g(x) = 0 \Rightarrow \cos(2\pi \sin x) = 0$

$\Rightarrow 2\pi \sin x = (2n+1)\dfrac{\pi}{2}, n \in I$

$\Rightarrow \sin x = \dfrac{2n+1}{4}$

$\Rightarrow \sin x = \pm\dfrac{1}{4}, \pm\dfrac{3}{4}$

$\Rightarrow Z = \left\{n\pi \pm \sin^{-1}\dfrac{1}{4}, n\pi \pm \sin^{-1}\dfrac{3}{4}, n \in I\right\}$

W = {x : g ′(x) = 0}

$g'(x) = 0 \Rightarrow -\sin(2\pi \sin x)(2\pi \cos x) = 0$

$\Rightarrow \sin(2\pi \sin x) = 0$ or $2\pi \cos x = 0$

$\Rightarrow 2\pi \sin x = n\pi$ or $x = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow \sin x = \dfrac{n}{2}$ or $x = (2n+1)\dfrac{n}{2}, n \in I$

$\Rightarrow \sin x = 0, \pm\dfrac{1}{2}, \pm 1$ or $x = (2n+1)\dfrac{\pi}{2}$

$W = \left\{n\pi, (2n+1)\dfrac{\pi}{2}, n\pi \pm \dfrac{\pi}{6}, n \in I\right\}$

option (1) (II) (Q) (T)

Q.2 Let f(x) = sin(π cos x) and g(x) = cos(2π sin x) be two functions defined for x > 0. Defined the following set whose elements are written in the increasing order :

X = {x : f(x) = 0}, Y = {x : f ′(x) = 0}

Z = {x : g(x) = 0}, W = {x : g ′(x) = 0}.

List –I contains the set X, Y, Z and W. List –II contains some information regarding these sets.

LIST-I		**LIST-II**	
(I)	X	**(P)**	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	**(Q)**	an arithmetic progression
(III)	Z	**(R)**	NOT an arithmetic progression
(IV)	W	**(S)**	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only CORRECT combination?

(1) (III), (P), (Q), (U) (2) (IV), (P), (R), (S) (3) (III), (R), (U) (4) (IV), (Q), (T)

Ans. **[2]**

Sol. Solution same as above Question (1)

Option (2) (IV) (P) (R) (S)

Q.3 Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions.

(i) centre of C_3 is collinear with the centres of C_1 and C_2,

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expressions given in the List-I whose values are given in List-II below :

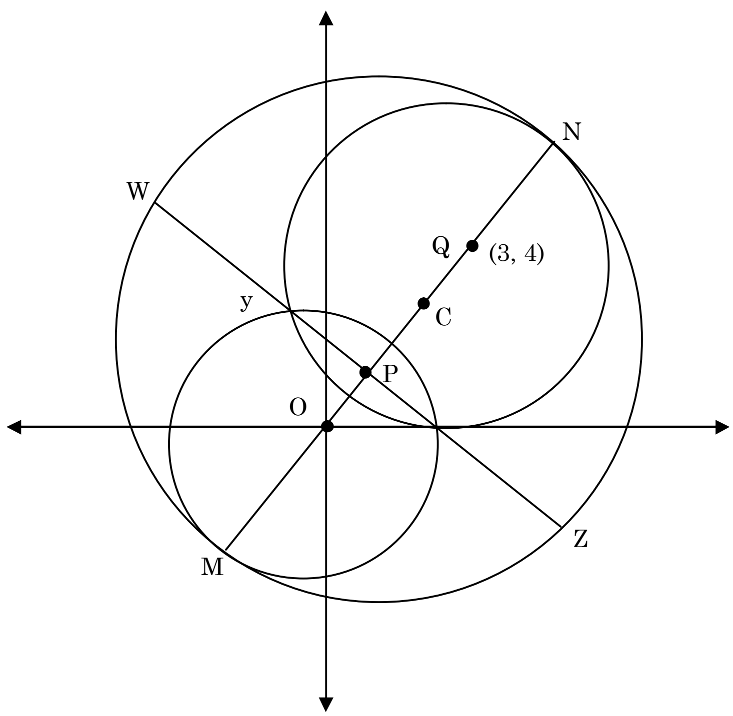
	LIST-I		LIST-II
(I)	$2h + k$	**(P)**	6
(II)	$\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	**(Q)**	$\sqrt{6}$
(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	**(R)**	$\dfrac{5}{4}$
(IV)	α	**(S)**	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

Which of the following is the only INCORRECT combination?

(1) (IV), (U) (2) (I), (P) (3) (IV), (S) (4) (III), (R)

Ans. **[3]**

Sol.



(I) Diameter of circle $C_3 \Rightarrow 2r = MN$

$= MO + OQ + QN$

$= 3 + \sqrt{3^2 + 4^2} + 4$

$= 12$

$\therefore\ r = 6$

Centre C of circle C_3 lies on $y = \frac{4}{3}x$.

Let $C\left(h, \frac{4}{3}h\right)$

OC = MC – OM

OC = 6 – 3 = 3

$\therefore\ \sqrt{h^2 + \frac{16}{9}h^2} = 3$

$\Rightarrow h = \frac{9}{5}$

$k = \frac{4}{3}h = \frac{12}{5}$

$\therefore\ 2h + k = \frac{18}{5} + \frac{12}{5} = 6$

(II) Equation of line ZW

$C_1 - C_2 = 0$

$\Rightarrow 3x + 4y = 9$

Distance of ZW from (0, 0)

$= \left|\frac{9}{\sqrt{9+16}}\right| = \frac{9}{5}$

Length of XY $= 2\sqrt{9 - \left(\frac{9}{5}\right)^2} = \frac{24}{5}$

Distance of ZW form C $\left(\frac{9}{5}, \frac{12}{5}\right)$

$\left|\frac{3\left(\frac{9}{5}\right) + 4\left(\frac{12}{5}\right) - 9}{\sqrt{9+16}}\right| = \frac{6}{5}$

Length of ZW $= 2\sqrt{36 - \left(\frac{6}{5}\right)^2} = \frac{24\sqrt{6}}{5}$

$\therefore\ \frac{\text{length of ZW}}{\text{length of XY}} = \sqrt{6}$

(III) Area of $\Delta MZN = \frac{1}{2}NM\left(\frac{1}{2}ZW\right) = \frac{72\sqrt{6}}{5}$

Area of $\Delta ZMW = \frac{1}{2}ZW(OM + OP)$

$$= \frac{1}{2}\frac{24\sqrt{6}}{5}\left(3+\frac{9}{5}\right) = \frac{288\sqrt{6}}{25}$$

$$\therefore \frac{\text{Area of } \Delta MZN}{\text{Area of } \Delta ZMW} = \frac{5}{4}$$

(IV) Slope of tangent to C_1 at M = $-\frac{3}{4}$

$\therefore$ Equation of tangent $y = mx - 3\sqrt{1+m^2}$

$$y = -\frac{3}{4}x - 3\sqrt{1+\frac{9}{16}}$$

$$\Rightarrow \quad y = -\frac{3x}{4} - \frac{15}{4} \quad \text{...(i)}$$

Tangent to parabola $x^2 = 8\alpha y$ is

$\Rightarrow y = mx - 2\alpha m^2$ ...(ii)

Compare (i) & (ii)

$m = -\frac{3}{4}$ and $2\alpha m^2 = +\frac{15}{4}$

$\Rightarrow 2\alpha\left(\frac{9}{16}\right) = \frac{15}{4}$

$\Rightarrow \alpha = \frac{10}{3}$

option (3) (IV) (S)

Q.4 Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions

(i) centre of C_3 is collinear with the centres of C_1 and C_2,

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expressions given in the List-I whose values are given in List-II below :

LIST-I		LIST-II	
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(II)	$\frac{\text{Length of ZW}}{\text{Length of XY}}$	**(Q)**	$\sqrt{6}$
(III)	$\frac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	**(R)**	$\frac{5}{4}$
(IV)	α	**(S)**	$\frac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\frac{10}{3}$

Which of the following is the only CORRECT combination?

(1) (II), (Q) (2) (II), (T) (3) (I), (S) (4) (I), (U)

Ans. **[1]**

Sol. Solution same as above Question (3)

Option (1); (II) (Q)