



JEE Advanced Exam 2019 (Paper & Solution)

Date : 27 / 05 / 2019

PAPER-1

PART-I (MATHEMATICS)

SECTION – 1 (Maximum Marks : 12)

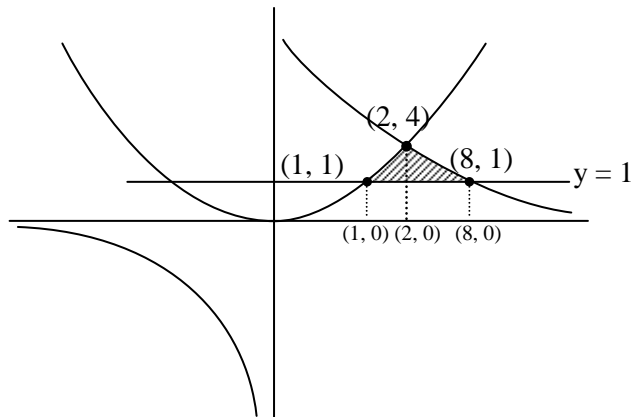
- This section contains **FOUR (04)** questions
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :
 - Full Marks : +3 If **ONLY** the correct option is chosen.
 - Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).
 - Negative Marks : -1 In all other cases.

Q.1 The area of the region $\{(x, y) : xy \leq 8, 1 \leq y \leq x^2\}$ is -

- (1) $8 \log_e 2 - \frac{14}{3}$ (2) $16 \log_e 2 - \frac{14}{3}$ (3) $16 \log_e 2 - 6$ (4) $8 \log_e 2 - \frac{7}{3}$

Ans. [2]

Sol.



$$\begin{aligned} \text{Required area} &= \int_1^2 x^2 dx + \int_2^8 \left(\frac{8}{x}\right) dx - (7) \\ &= \left[\frac{x^3}{3}\right]_1^2 + 8(\ln x)_2^8 - 7 \end{aligned}$$

$$\begin{aligned}
 &= \frac{8}{3} - \frac{1}{3} + 8(\ln 8 - \ln 2) - 7 \\
 &= \frac{7}{3} + 8(2 \ln 2) - 7 \\
 &= -\frac{14}{3} + 16 \ln 2
 \end{aligned}$$

Q.2 Let $M = \begin{bmatrix} \sin^4 \theta & -1 - \sin^2 \theta \\ 1 + \cos^2 \theta & \cos^4 \theta \end{bmatrix} = \alpha I + \beta M^{-1}$, where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and I is the

2×2 identity matrix. If α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$, then value of $\alpha^* + \beta^*$ is -

(1) $-\frac{37}{16}$ (2) $-\frac{31}{16}$ (3) $-\frac{17}{16}$ (4) $-\frac{29}{16}$

Ans. [4]

Sol. $m = \sin^4 \theta \cos^4 \theta + (1 + \sin^2 \theta)(1 + \cos^2 \theta)$

$$m = 2 + \sin^4 \theta \cos^4 \theta + \sin 2\theta \cos^2 \theta$$

$$\begin{bmatrix} \sin^4 \theta & -1 - \sin^2 \theta \\ 1 + \cos^2 \theta & \cos^4 \theta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \frac{\beta}{|M|} \begin{bmatrix} \cos^4 \theta & 1 + \sin^2 \theta \\ -1 - \cos^2 \theta & \sin^4 \theta \end{bmatrix}$$

$$\sin^4 \theta = \alpha + \frac{\beta}{|M|} \cos^4 \theta \quad \dots(1)$$

$$-1 - \sin^2 \theta = \frac{\beta}{|M|} (1 + \sin^2 \theta) \quad \dots(2)$$

$$1 + \cos^2 \theta = \frac{\beta}{|M|} (-1 - \cos^2 \theta) \quad \dots(3)$$

$$\cos^4 \theta = \alpha + \frac{\beta}{|M|} \sin^4 \theta \quad \dots(4)$$

From eqⁿ (3) $\beta = -|M|$

$$\beta = -(\sin^4 \theta \cos^4 \theta + \sin^2 \theta \cos^2 \theta + 2)$$

$$\beta = -(t^2 + t + 2) \quad t = \sin^2 \theta \cos^2 \theta$$

$$t = \frac{1}{4}(\sin 2\theta)^2$$

$$0 \leq t \leq \frac{1}{4}$$

$$\beta_{\min} \text{ at } t = \frac{1}{4} \quad \beta_{\min} = -\left(\frac{1}{16} + \frac{1}{4} + 2\right)$$

$$\beta_{\min} = -\frac{1 + 4 + 32}{16}$$

$$\beta_{\min} = -\frac{37}{16}$$

From (1)

$$\sin^4 \theta = \alpha - \cos^4 \theta$$

$$\alpha = \sin^4\theta + \cos^4\theta$$

$$\alpha = 1 - \frac{1}{2}(\sin^2 2\theta)$$

$$\alpha_{\min} = \frac{1}{2}$$

$$\alpha_{\min} + \beta_{\min} = \frac{-37}{16} + \frac{1}{2} = \frac{-29}{16} \quad \text{Option (4) is correct}$$

Q.3 A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

- (1) $-3 \leq m < -1$ (2) $2 \leq m < 4$ (3) $4 \leq m < 6$ (4) $6 \leq m < 8$

Ans. [2]

Sol. Coordinate of mid point of chord

$$\left(-\frac{3}{5}, -\frac{3m}{5} + 1 \right)$$

Slope of line joining mid point of chord with centre $(3, -2)$

$$= \frac{-\frac{3m}{5} + 1 + 2}{-\frac{3}{5} - 3} = \frac{-3m + 15}{-18}$$

Now this line is perpendicular to given chord

$$\left(\frac{-3m + 15}{-18} \right) (m) = -1$$

$$-3m^2 + 15m = 18$$

$$3m^2 - 15m + 18 = 0$$

$$m^2 - 5m + 6 = 0$$

$$(m - 2)(m - 3) = 0$$

$$m = 2, m = 3$$

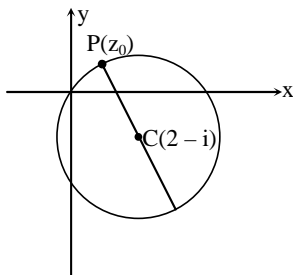
option (2) is correct $2 \leq m \leq 4$

Q.4 Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{ \frac{1}{|z - 1|} : z \in S \right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is -

- (1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$ (3) $-\frac{\pi}{2}$ (4) $\frac{3\pi}{4}$

Ans. [3]

Sol.





$$|z - (2 - i)| \geq \sqrt{5}$$

for $|z_0 - 1|$ to be minimum

$z_0 = x_0 + iy_0$ is at point P as shown in figure

$$\arg \left(\frac{4 - (z_0 + \bar{z}_0)}{z_0 - \bar{z}_0 + 2i} \right) = \arg \left(\frac{4 - 2x}{2iy + 2i} \right)$$

$$= \arg \left(\frac{-i(2-x)}{y+2} \right)$$

$$= \arg(-i\lambda) = -\frac{\pi}{2} \quad \text{option (3) is correct}$$

SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: +4	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct.
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: 0	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	: -1	In all other cases.
- For example, in a question, if (A), (B) and (D) are the **ONLY** three options corresponding to correct answers, then
 - choosing **ONLY** (A), (B) and (D) will get +4 marks;
 - choosing **ONLY** (A) and (B) will get +2 marks;
 - choosing **ONLY** (A) and (D) will get +2 marks;
 - choosing **ONLY** (B) and (D) will get +2 marks;
 - choosing **ONLY** (A) will get +1 marks;
 - choosing **ONLY** (B) will get +1 marks;
 - choosing **ONLY** (D) will get +1 marks;
 - choosing no option (i.e. the question is unanswered) will get 0 marks; and
 - choosing any other combination of options will get -1 mark.

Q.1 Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n , define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1$$

$$b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, n \geq 2.$$

Then which of the following options is/are correct ?

(1) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

(2) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $a_1 + a_2 + a_3 + \dots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(4) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Ans. [1,2,3]

Sol. $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}$$

$$(1) \quad b_n = a_{n-1} + a_{n-1}$$

$$b_n = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

$$b_n = \frac{\alpha^{n-1} - \beta^{n-1} + \alpha^{n-1} - \beta^{n-1}}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}(\alpha^2 + 1) - \beta^{n-1}(\beta^2 + 1)}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}(\alpha + 2) - \beta^{n-1}(\beta + 2)}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1} \left(\frac{5 + \sqrt{5}}{2} \right) - \beta^{n-1} \left(\frac{5 - \sqrt{5}}{2} \right)}{(\alpha - \beta)}$$

$$b_n = \frac{\sqrt{5} \alpha^{n-1} \left(\frac{\sqrt{5} + 1}{2} \right) - \sqrt{5} \beta^{n-1} \left(\frac{\sqrt{5} - 1}{2} \right)}{(\alpha - \beta)}$$

$$b_n = \frac{\sqrt{5} \alpha^{n-1} (\alpha) - \sqrt{5} \beta^{n-1} (-\beta)}{\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}}$$

$$b_n = \frac{\sqrt{5} \alpha^n + \sqrt{5} \beta^n}{\sqrt{5}}$$

$$b_n = \alpha^n + \beta^n$$

Option (1) is correct

$$(2) \quad \sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{(\alpha - \beta) 10^n}$$

$$= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left(\frac{\alpha}{10} \right)^n - \left(\frac{\beta}{10} \right)^n$$

$$= \frac{1}{\sqrt{5}} \left[\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} \right]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\alpha(10 - \beta) - \beta(10 - \alpha)}{(10 - \alpha)(10 - \beta)} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{10\alpha - \alpha\beta - 10\beta + \alpha\beta}{100 - 10(\alpha + \beta) + \alpha\beta} \right)$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{5}} \frac{10(\alpha - \beta)}{100 - 10(\alpha + \beta) + \alpha\beta} \\
 &= \frac{10}{100 - 10 - 1} = \frac{10}{89}
 \end{aligned}$$

option (2) is correct

$$\begin{aligned}
 (3) \quad &a_1 + a_2 + a_3 + \dots + a_n \\
 &= \sum_{i=1}^n a_i = \sum_{i=1}^n \frac{\alpha^i - \beta^i}{\alpha - \beta} \\
 &= \frac{1}{(\alpha - \beta)} \left(\frac{\alpha(1 - \alpha^n)}{(1 - \alpha)} - \frac{\beta(1 - \beta^n)}{(1 - \beta)} \right) \\
 &= \frac{(\alpha + 1)(1 - \alpha^n) - (\beta + 1)(1 - \beta^n)}{(1 - \alpha)(1 - \beta)(\alpha - \beta)} \\
 &= \frac{\alpha^2 - \alpha^{n+2} - \beta^2 + \beta^{n+2}}{(1 - \alpha)(1 - \beta)(\alpha - \beta)} \\
 &= \frac{\sqrt{5} + \beta^{n+2} - \alpha^{n+2}}{(\beta - \alpha)} = -1 + a_{n+2}
 \end{aligned}$$

option (3) is correct

$$\begin{aligned}
 (4) \quad &\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{\alpha}{10} \right)^n + \sum_{n=1}^{\infty} \left(\frac{\beta}{10} \right)^n \\
 &= \frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} = \frac{\alpha}{10 - \alpha} + \frac{\beta}{10 - \beta} \\
 &= \frac{10(\alpha + \beta) - 2\alpha\beta}{100 - 10(\alpha + \beta) + \alpha\beta} = \frac{10 + 2}{89} = \frac{12}{89}
 \end{aligned}$$

option (4) is wrong

Q.2 Let L_1 and L_2 denote the lines $\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k})$, $\lambda \in \mathbb{R}$ and $\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k})$, $\mu \in \mathbb{R}$ respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ?

$$(1) \quad \vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), \quad t \in \mathbb{R}$$

$$(2) \quad \vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), \quad t \in \mathbb{R}$$

$$(3) \quad \vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), \quad t \in \mathbb{R}$$

$$(4) \quad \vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), \quad t \in \mathbb{R}$$

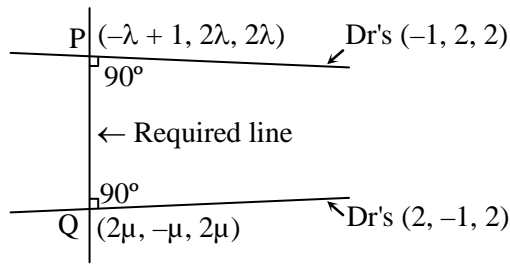
Ans. [1,3,4]

Sol. Line (1): $\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k})$, $\lambda \in \mathbb{R}$

Line (2): $\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k})$, $\mu \in \mathbb{R}$

$$\text{Line (1)} \quad \frac{x-1}{-1} = \frac{y-0}{2} = \frac{z-0}{2} = \lambda$$

$$\text{Line (2)} \quad \frac{x-0}{2} = \frac{y-0}{-1} = \frac{z-0}{2} = \mu$$



Dr's of line PQ $(-\lambda - 2\mu + 1, 2\lambda + \mu, 2\lambda - 2\mu)$

Let first line parallel to $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Let second line parallel to $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \hat{i}(6) - \hat{j}(-2 - 4) + \hat{k}(1 - 4) \\ &= 6\hat{i} + 6\hat{j} - 3\hat{k} \end{aligned}$$

So that Dr's of required line $(2, 2, -1)$

$$\text{Now } \frac{-\lambda - 2\mu + 1}{2} = \frac{2\lambda + \mu}{2} = \frac{2\lambda - 2\mu}{-1}$$

$$\begin{array}{l|l} -\lambda - 2\mu + 1 = 2\lambda + \mu & -2\lambda - \mu = 4\lambda - 4\mu \\ 3\lambda + 3\mu = 1 \quad \dots(1) & 6\lambda = 3\mu \\ & \mu = 2\lambda \quad \dots(2) \end{array}$$

From (1) & (2)

$$\lambda = \frac{1}{9}$$

$$\mu = \frac{2}{9}$$

$$\text{Point P } \left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9} \right)$$

$$\text{Point Q } \left(\frac{4}{9}, \frac{-2}{9}, \frac{4}{9} \right)$$

Now equation of required line

$$\vec{r} = \left(\frac{8}{9}\hat{i} + \frac{2}{9}\hat{j} + \frac{2}{9}\hat{k} \right) + t(2\hat{i} + 2\hat{j} - \hat{k}) \rightarrow \text{option (3) correct}$$

$$\vec{r} = \left(\frac{4}{9}\hat{i} - \frac{2}{9}\hat{j} + \frac{4}{9}\hat{k} \right) + t(2\hat{i} + 2\hat{j} - \hat{k}) \rightarrow \text{option (4) correct}$$

option 2 is wrong

option (1) is correct

Q.3 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \leq x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \geq 3 \end{cases}$

Then which of the following options is/are correct ?

- (1) f is increasing on $(-\infty, 0)$ (2) f is onto
(3) f' has a local maximum at $x = 1$ (4) f' is NOT differentiable at $x = 1$

Ans. [2,3,4]

Sol.

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \leq x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \leq x < 3 \\ (x-2)\log_e(x-2) - x + \frac{10}{3} & x \geq 3 \end{cases}$$

$$f(x) = \begin{cases} 5(x+1)^4 - 2 & x < 0 \\ 2x - 1 & 0 \leq x < 1 \\ 2x^2 - 8x + 7 & 1 \leq x < 3 \\ \ln(x-2) & x \geq 3 \end{cases}$$

$x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1$ takes value between $-\infty$ to 1

also $(x-2) \ln(x-2) - x + \frac{10}{3}$ takes value between $\frac{1}{3}$ to ∞

So range of $f(x)$ is \mathbb{R} .

option (1) is correct

$f''(1^-) = 2$ and $f''(1^+) = -4$

so $f'(x)$ is not diff. at $x = 1$

so option (2) is correct

$f'(x)$ has local maxima at $x = 1$

so option (3) is correct

Q.4 Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\text{adj } M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the

following options is/are correct ?

(1) If $M = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$ (2) $a + b = 3$

(3) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$ (4) $\det(\text{adj } M^2) = 81$



Ans. [1,2,3]

Sol. $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ $\text{adj } M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$$2 - 3b = -1$$

$$3b = 3$$

$$b = 1$$

$$3 - 2a = -1$$

$$-2a = -4$$

$$a = 2$$

Option (1)

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\beta + 2\gamma = 1$$

$$\alpha + 2\beta + 3\gamma = 2$$

$$3\alpha + \beta + \gamma = 3$$

Solve it

$$\alpha = 1, \beta = -1, \gamma = 1$$

$$\alpha - \beta + \gamma = 1 + 1 + 1 = 3$$

option (1) correct

Option (2)

$$a + b = 1 + 2 = 3$$

$$\text{Option (3) } (\text{adj } M)^{-1} \text{ adj } M^{-1}$$

$$= 2 \text{ adj } (M^{-1})$$

$$= 2 \left(-\frac{M}{2} \right)$$

$$= -M$$

Option (4)

$$|\text{adj } M^2| = |M|^2 = |M|^4 = 16$$

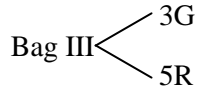
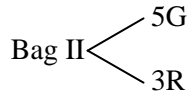
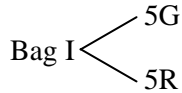
Q.5 There are three bags B_1, B_2 and B_3 . The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}, \frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

(1) Probability that the selected bag is B_3 , given that the chosen ball is green, equals $\frac{5}{13}$

(2) Probability that the chosen ball is green, given that the selected bag is B_3 , equals $\frac{3}{8}$

(3) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(4) Probability that the chosen ball is green equals $\frac{39}{80}$

Ans. [2 & 4]
Sol.


$$\text{Option (1)} \quad P\left(\frac{B_3}{G}\right) = \frac{P(B_3 \cap G)}{P(G)} = \frac{\frac{4}{10} \times \frac{3}{8}}{\frac{12}{39}} = \frac{12}{39} = \frac{4}{13}$$

$$\text{Option (2)} \quad P\left(\frac{G}{B_3}\right) = \frac{P(G \cap B_3)}{P(B_3)} = \frac{P(G)P(B_3)}{P(B_3)} = P(G) = \frac{3}{8}$$

$$\text{Option (3)} \quad P(B_3 \cap G) = P(B_3) P(G) = \frac{4}{10} \times \frac{3}{8} = \frac{3}{20}$$

$$\begin{aligned} \text{Option (4)} \quad P(G) &= P(B_1) P\left(\frac{G}{B_1}\right) + P(B_2) P\left(\frac{G}{B_2}\right) + P(B_3) P\left(\frac{G}{B_3}\right) \\ &= \frac{3}{10} \times \frac{5}{10} + \frac{3}{10} \times \frac{5}{8} + \frac{4}{10} \times \frac{3}{8} \\ &= \frac{3}{20} + \frac{15}{80} + \frac{12}{80} \\ &= \frac{12+15+12}{80} = \frac{39}{80} \end{aligned}$$

Option (2) & (4) correct

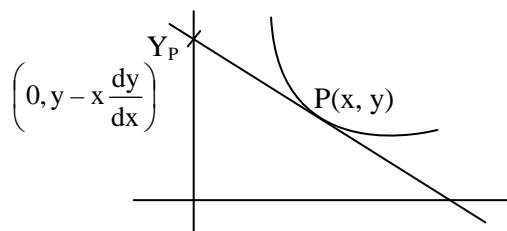
Q.6 Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to Γ at a point P intersect the y -axis at Y_P . If PY_P has length l for each point P on Γ , then which of the following options is/are correct ?

(1) $xy' + \sqrt{1-x^2} = 0$

(2) $y = \log_e \left(\frac{1 + \sqrt{1-x^2}}{x} \right) - \sqrt{1-x^2}$

(3) $xy' - \sqrt{1-x^2} = 0$

(4) $y = -\log_e \left(\frac{1 + \sqrt{1-x^2}}{x} \right) + \sqrt{1-x^2}$

Ans. [1,2,3,4]
Sol.


equation of tangent

$$(Y - y) = \frac{dy}{dx} (X - x)$$

Put $X = 0$

$$Y = y - x \cdot \frac{dy}{dx}$$

Distance between $PY_P = 1$

$$\sqrt{x^2 + x^2 \left(\frac{dy}{dx} \right)^2} = 1$$

$$x^2 + x^2 \left(\frac{dy}{dx} \right)^2 = 1$$

$$x^2 \left(\frac{dy}{dx} \right)^2 = 1 - x^2$$

$$x \frac{dy}{dx} = \pm \sqrt{1 - x^2}$$

$$x \frac{dy}{dx} = \sqrt{1 - x^2}$$

$$\int dy = \int \frac{\sqrt{1 - x^2}}{x} dx$$

put $x = \sin \theta$

$$dx = \cos \theta d\theta$$

$$y = \int \frac{\cos^2 \theta}{\sin \theta} d\theta$$

$$y = \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta$$

$$y = \int (\operatorname{cosec} \theta - \sin \theta) d\theta$$

$$y = \ell n (\operatorname{cosec} \theta - \cot \theta) + \cos \theta + c$$

$$y = \ell n \left(\frac{1}{x} - \frac{\sqrt{1 - x^2}}{x} \right) + \sqrt{1 - x^2} + c$$

$$y = \ell n \left(\frac{1 - \sqrt{1 - x^2}}{x} \right) + \sqrt{1 - x^2} + c$$

$$y = \ell n \left(\frac{x^2}{1 + \sqrt{1 - x^2}} \times \frac{1}{x} \right) + \sqrt{1 - x^2} + c$$

$$y = -\ell n \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) + \sqrt{1 - x^2} + c$$

$$x \frac{dy}{dx} = -\sqrt{1 - x^2}$$

$$dy = -\int \frac{\sqrt{1 - x^2}}{x} dx$$

Integrate

$$y = \ell n \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2} + c$$

$$f(1) = 0, \quad c = 0$$

$$y = \ell n \left(\frac{1 + \sqrt{1 - x^2}}{x} \right) - \sqrt{1 - x^2}$$



Now $F(1) = 0$

$c = 0$

$$y = -\ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$$

Q.7 Define the collection $\{E_1, E_2, E_3, \dots\}$ of ellipses and $\{R_1, R_2, R_3, \dots\}$ of rectangles as follows

$$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1;$$

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

$$E_n : \text{ellipse } \frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1 \text{ of largest area inscribed in } R_{n-1}, n > 1;$$

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in $E_n, n > 1$

Then which of the following options is/are correct ?

(1) $\sum_{n=1}^N (\text{area of } R_n) < 24$, for each positive integer N

(2) The eccentricities of E_{18} and E_{19} are NOT equal

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

Ans. [1, 3]

Sol. Area maximum when $\theta = 45^\circ$

	a	b
E_1	3	2
E_2	$\frac{3}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$
E_3	$\frac{3}{(\sqrt{2})^2}$	$\frac{2}{(\sqrt{2})^2}$

⋮
⋮

(A) $E_1 + E_2 + \dots + E_m$

$$\text{when } m \rightarrow \infty \frac{2ab}{1 - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = 4ab = 4 \cdot 3 \cdot 2 = 24$$

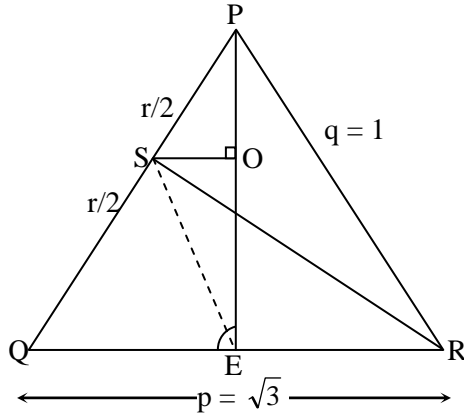
(B) Length of L.R. = $\frac{2b^2}{a} = \frac{2 \cdot 4 \cdot 24}{2 \cdot 8 \cdot 3} = \frac{1}{6}$

(C) Distance between focus and centre of ellipse of $I_9 = \frac{3}{2^4} \cdot \frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{16}$

Q.8 In a non-right-angled triangle ΔPQR , let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S , the perpendicular from P meets the side QR at E and RS and PE intersect at O . If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

- (1) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$ (2) Length of $OE = \frac{1}{6}$
 (3) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$ (4) Length of $RS = \frac{\sqrt{7}}{2}$

Ans. [1,2,4]
Sol.



$$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2(1)$$

$$\Rightarrow \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2}$$

$$\angle P = 60^\circ \text{ or } 120^\circ$$

$$\text{and } \angle Q = 30^\circ \text{ or } 150^\circ$$

$$\because \angle P + \angle Q < 180^\circ \text{ and } \neq 90^\circ$$

$$\angle P = 120^\circ \text{ \& } \angle Q = 30^\circ \text{ \& } \angle R = 30^\circ$$

$$\frac{r}{\sin R} = 2 \Rightarrow r = 1$$

$$\text{Medians } RS = \frac{1}{2} \sqrt{2p^2 + 2q^2 - r^2} = \frac{1}{2} \sqrt{6 + 2 - 1} = \frac{\sqrt{7}}{2}$$

$$\text{inradius} = \frac{2\Delta}{p+q+r} = \frac{2pqr}{4 \times 1} = \frac{1}{2} \left(\frac{1 \times 1 \times \sqrt{3}}{1+1+\sqrt{3}} \right) = \frac{\sqrt{3}}{2} \left(\frac{2-\sqrt{3}}{1} \right)$$

$$\Rightarrow \frac{1}{2} \times \sqrt{3} \times PE = \frac{pqr}{4(1)} \text{ (equal area of } \Delta)$$

$$PE = \frac{1 \times 1 \times \sqrt{3}}{4} \times \frac{2}{\sqrt{3}} = \frac{1}{2}$$

$$\Rightarrow OE = \frac{2(\text{area of } \Delta OPQ)}{QR} = \frac{2 \times \frac{1}{3} \left(\frac{1}{2} \times 1 \times \sqrt{3} \sin 30^\circ \right)}{\sqrt{3}} = \frac{1}{6}$$

SECTION – 3 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme.
Full Marks : +3 If **ONLY** the correct numerical value is entered.
Zero Marks : 0 In all other cases.

Q.1 Three lines are given by $\vec{r} = \lambda \hat{i}$, $\lambda \in \mathbb{R}$; $\vec{r} = \mu(\hat{i} + \hat{j})$, $\mu \in \mathbb{R}$ and $\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k})$, $\nu \in \mathbb{R}$. Let the lines cut the plane $x + y + z = 1$ at the points A, B and C respectively. If the area of the triangle ABC is Δ , then the value of $(6\Delta)^2$ equals _____

Ans. [0.75]

Sol. $A(\lambda, 0, 0)$

$B(\mu, \mu, 0)$

$C(\nu, \nu, \nu)$

A, B & C satisfies $x + y + z = 1$

$$\therefore \lambda = 1$$

$$\mu = \frac{1}{2}$$

$$\nu = \frac{1}{3}$$

$$\therefore A(1, 0, 0), B\left(\frac{1}{2}, \frac{1}{2}, 0\right), C\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\vec{AB} = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + 0\hat{k}$$

$$\vec{AC} = -\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$$

$$\Delta = \frac{1}{2} |\vec{AB} \times \vec{AC}| \Rightarrow \frac{1}{2} \left\| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ -1/2 & 1/2 & 0 \\ -2/3 & 1/3 & 1/3 \end{array} \right\|$$

$$\Delta = \frac{\sqrt{3}}{2 \times 6}$$

$$\therefore 36\Delta^2 = \frac{3}{4} = 0.75$$

Q.2 If $I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{\sin x})(2 - \cos 2x)}$, then $27I^2$ equals ____

Ans. [4.00]

Sol. $I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{\sin x})(2 - \cos 2x)}$

$$2I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(2 - \cos 2x)}$$

$$I = \frac{2}{\pi} \int_0^{\pi/4} \frac{dx}{3 - 2\cos^2 x}$$

$$I = \frac{2}{\pi} \int_0^{\pi/4} \frac{\sec^2 x}{1 + 3\tan^2 x} dx$$

$$\sqrt{3} \tan x = t$$

$$\sec^2 x dx = \frac{dt}{\sqrt{3}}$$

$$I = \frac{2}{\pi} \times \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{dt}{1+t^2}$$

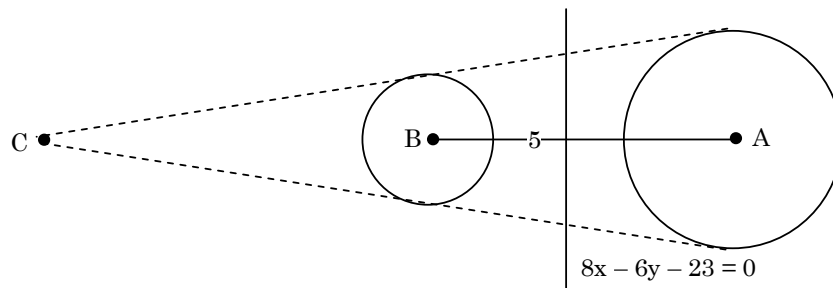
$$I = \frac{2}{\sqrt{3}\pi} \times \frac{\pi}{3} = \frac{2}{3\sqrt{3}}$$

$$27I^2 = 27 \times \frac{4}{9 \times 3} = 4$$

Q.3 Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is ____.

Ans. [10.00]

Sol.



$$AL = \left| \frac{16 - 18 - 23}{10} \right| = \frac{5}{2}$$

$$\frac{CB}{CA} = \frac{1}{2}$$

$$\frac{CA - 5}{CA} = \frac{1}{2}$$

$$CA = 10$$

Q.4 Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$ equals ____

Ans. [3.00]

Sol. $|a + b\omega + c\omega^2|^2 = a^2 + b^2 + c^2 - ab - bc - ca$
 $= \frac{1}{2} [(a - b)^2 + (b - c)^2 + (c - a)^2]$

It will be minimum when a, b, c are consecutive integers.
So, minimum value is '3'

Q.5 Let $AP(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then $a + d$ equals ____

Ans. [157.00]

Sol. First series $\{1, 4, 7, 10, 13, \dots\} \rightarrow c.d = 3$
Second series $\{2, 7, 12, 17, \dots\} \rightarrow c.d = 5$
Third series $\{3, 10, 17, 24, \dots\} \rightarrow c.d = 7$
First common term of these series is 52
New common difference of new series is
L.C.M of $(3, 5, 7) = 105$
 $\therefore a = 52$
 $d = 105$
 $a + d = 157$

Q.6 Let S be the sample space of all 3×3 matrices with entries from the set $\{0, 1\}$. Let the events E_1 and E_2 be given by $E_1 = \{A \in S : \det A = 0\}$ and $E_2 = \{A \in S : \text{sum of entries of } A \text{ is } 7\}$. If a matrix is chosen at random from S , then the conditional probability $P(E_1|E_2)$ equals ____

Ans. [0.50]

Sol. In matrix A there are 7 ones and 2 zeros
 \therefore Sum of elements of $A = 7$

Number of such matrices = 9C_2

Out of all such matrices E_1 will be those when both zeros lie in the same row or in the same column

eq.
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$n(E_1 \cap E_2) = 2 \times {}^3C_2 \times {}^3C_2 = 18$$

$$\text{So, } n\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{16} = \frac{1}{2} = 0.50$$