1. Let \( M = \begin{bmatrix} \sin^4 \theta & -1 - \sin^2 \theta \\ 1 + \cos^2 \theta & \cos^4 \theta \end{bmatrix} = \alpha I + \beta M^{-1} \),

where \( \alpha = \alpha(\theta) \) and \( \beta = \beta(\theta) \) are real number, and I is the 2 \( \times \) 2 identity matrix. If

\[ \alpha^* \] is the minimum of the set \( \{ \alpha(\theta) : \theta \in [0, 2\pi) \} \) and

\[ \beta^* \] is the minimum of the set \( \{ \beta(\theta) : \theta \in [0, 2\pi) \} \),

then the value of \( \alpha^* + \beta^* \) is

(1) \( \frac{37}{16} \)

(2) \( \frac{29}{16} \)

(3) \( \frac{31}{16} \)

(4) \( \frac{17}{16} \)

Ans. (2)

Sol. Given \( M = \alpha I + \beta M^{-1} \)

\( \Rightarrow M^2 - \alpha M - \beta I = 0 \)

By putting values of M and \( M^2 \), we get

\[ \alpha(\theta) = 1 - 2\sin^2 \theta \cos^2 \theta = 1 - \frac{\sin^2 2\theta}{2} \geq \frac{1}{2} \]

Also, \( \beta(\theta) = -(\sin^4 \theta \cos^4 \theta + (1 + \cos^2 \theta)(1 + \sin^2 \theta)) \)

\[ = -(\sin^4 \theta \cos^4 \theta + 1 + \cos^2 \theta + \sin^2 \theta + \sin^2 \theta \cos^2 \theta) \]

\[ = -(t^2 + t + 2), \quad t = \frac{\sin^2 2\theta}{4} \in \left[ 0, \frac{1}{4} \right] \]

\[ \Rightarrow \beta(\theta) \geq -\frac{37}{16} \]
2. A line \( y = mx + 1 \) intersects the circle \((x - 3)^2 + (y + 2)^2 = 25\) at the points P and Q. If the midpoint of the line segment PQ has x-coordinate \( \frac{3}{5} \), then which one of the following options is correct?

(1) \( 6 \leq m < 8 \)  
(2) \( 2 \leq m < 4 \)  
(3) \( 4 \leq m < 6 \)  
(4) \( -3 \leq m < -1 \)

Ans. (2)

Sol.

So, \( m = \frac{-3m + 3}{\frac{3}{5} - \frac{3}{5}} = -1 \)

\( \Rightarrow m^2 - 5m + 6 = 0 \)
\( \Rightarrow m = 2, 3 \)

3. Let \( S \) be the set of all complex numbers \( z \) satisfying \( |z - 2 + i| \geq \sqrt{5} \). If the complex number \( z_0 \) is such that \( \frac{1}{|z_0 - 1|} \) is the maximum of the set \( \left\{ \frac{1}{|z - 1|} : z \in S \right\} \), then the principal argument of \( \frac{4 - z_0 - z_0}{z_0 - z_0 + 2i} \) is

(1) \( \frac{\pi}{4} \)  
(2) \( \frac{-\pi}{2} \)  
(3) \( \frac{3\pi}{4} \)  
(4) \( \frac{\pi}{2} \)

Ans. (2)

Sol.

\[ \arg \left( \frac{4 - (z_0 - z_0)}{(z_0 - z_0) + zi} \right) \]
\[ = \arg \left( \frac{4 - 2 \text{Re} z_0}{2i \text{Im} z_0 + 2i} \right) = \arg \left( \frac{2 - \text{Re} z_0}{1 + \text{Im} z_0} i \right) \]
\[ = \arg \left( \frac{2 - \text{Re} z_0}{1 + \text{Im} z_0} i \right) \]
\[ = \arg(-ki) ; k > 0 \quad (\text{as Re} z_0 < 2 \& \text{Im} z_0 > 0) \]
\[ = \frac{-\pi}{2} \]
4. The area of the region \( \{(x, y) : xy \leq 8, 1 \leq y \leq x^2\} \) is

\[
(1) \, 8 \log_e 2 - \frac{14}{3} \quad (2) \, 16 \log_e 2 - \frac{14}{3} \quad (3) \, 16 \log_e 2 - 6 \quad (4) \, 8 \log_e 2 - \frac{7}{3}
\]

Ans. (2)

\[
\text{Sol.}
\]

For intersection, \( \frac{8}{y} = \sqrt{y} \Rightarrow y = 4 \)

Hence, required area = \( \int_{1}^{4} \left( \frac{8}{y} - \sqrt{y} \right) dy \)

\[
= \left[ 8 \ln y - \frac{2}{3} y^{3/2} \right]_{1}^{4} = 16 \ln 2 - \frac{14}{3}
\]

Remark: The question should contain the phrase "area of the bounded region in the first quadrant". Because, in the 2\textsuperscript{nd} quadrant, the region above the line \( y = 1 \) and below \( y = x^2 \), satisfies the region, which is unbounded.

SECTION-2: (Maximum Marks: 32)

- This section contains EIGHT (08) questions.
- Each question has FOUR options. ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:
  - Full Marks: +4 If only (all) the correct option(s) is (are) chosen.
  - Partial Marks: +3 If all the four options are correct but ONLY three options are chosen.
  - Partial Marks: +2 If three or more options are correct but ONLY two options are chosen and both of which are correct.
  - Partial Marks: +1 If two or more options are correct but ONLY one option is chosen and it is a correct option.
  - Zero Marks: 0 If none of the options is chosen (i.e. the question is unanswered).
  - Negative Marks: -1 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  - choosing ONLY (A), (B) and (D) will get +4 marks;
  - choosing ONLY (A) and (B) will get +2 marks;
  - choosing ONLY (A) and (D) will get +2 marks;
  - choosing ONLY (B) and (D) will get +2 marks;
  - choosing ONLY (A) will get +1 marks;
  - choosing ONLY (B) will get +1 marks;
  - choosing ONLY (D) will get +1 marks;
  - choosing no option (i.e. the question is unanswered) will get 0 marks, and choosing any other combination of options will get –1 mark.
1. There are three bags \( B_1, B_2 \) and \( B_3 \). The bag \( B_1 \) contains 5 red and 5 green balls, \( B_2 \) contains 3 red and 5 green balls, and \( B_3 \) contains 5 red and 3 green balls. Bags \( B_1, B_2 \) and \( B_3 \) have probabilities \( \frac{3}{10}, \frac{3}{10} \) and \( \frac{4}{10} \) respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct?

1. Probability that the selected bag is \( B_3 \) and the chosen ball is green equals \( \frac{3}{10} \)

2. Probability that the chosen ball is green equals \( \frac{39}{80} \)

3. Probability that the chosen ball is green, given that the selected bag is \( B_3 \), equals \( \frac{3}{8} \)

4. Probability that the selected bag is \( B_3 \), given that the chosen balls is green, equals \( \frac{5}{13} \)

Ans. (2,3)

Sol.

<table>
<thead>
<tr>
<th>Ball</th>
<th>Balls composition</th>
<th>( P(B_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_1 )</td>
<td>5R + 5G</td>
<td>( \frac{3}{10} )</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>3R + 5G</td>
<td>( \frac{3}{10} )</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>5R + 3G</td>
<td>( \frac{4}{10} )</td>
</tr>
</tbody>
</table>

(1) \[ P(B_3 \cap G) = P \left( \frac{G_1}{B_3} \right) P(B_3) \]
\[ = \frac{3}{8} \times \frac{4}{10} = \frac{3}{20} \]

(2) \[ P(G) = P \left( \frac{G_1}{B_1} \right) P(B_1) + P \left( \frac{G_1}{B_2} \right) P(B_2) + P \left( \frac{G_1}{B_3} \right) P(B_3) \]
\[ = \frac{3}{20} + \frac{3}{16} + \frac{3}{20} = \frac{39}{80} \]

(3) \[ P \left( \frac{G}{B_3} \right) = \frac{3}{8} \]

(4) \[ P \left( \frac{B_3}{G} \right) = \frac{P(G \cap B_3)}{P(G)} = \frac{3/20}{39/80} = \frac{4}{13} \]
2. Define the collections \{E_1, E_2, E_3, \ldots\} of ellipses and \{R_1, R_2, R_3, \ldots\} of rectangles as follows:

\[ E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1 ; \]
\[ R_1 : \text{rectangle of largest area, with sides parallel to the axes, inscribed in } E_1 ; \]
\[ E_n : \text{ellipse } \frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1 \text{ of largest area inscribed in } R_{n-1}, \ n > 1 ; \]
\[ R_n : \text{rectangle of largest area, with sides parallel to the axes, inscribed in } E_n, \ n > 1. \]

Then which of the following options is/are correct?

(1) The eccentricities of \(E_{18}\) and \(E_{19}\) are NOT equal

(2) The distance of a focus from the centre in \(E_9\) is \(\frac{\sqrt{55}}{32}\)

(3) The length of latus rectum of \(E_9\) is \(\frac{1}{6}\)

(4) \(\sum_{n=1}^{N} (\text{area of } R_n) < 24\), for each positive integer \(N\)

Ans. (3,4)

Sol.

Area of \(R_1 = 3\sin 2\theta\) ; for this to be maximum

\[ \Rightarrow 0 = \frac{\pi}{4} \Rightarrow \left( \frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right) \]

Hence for subsequent areas of rectangles \(R_n\) to be maximum the coordinates will be in GP with common ratio \(r = \frac{1}{\sqrt{2}} \Rightarrow a_n = \frac{3}{(\sqrt{2})^{n-1}} \); \(b_n = \frac{3}{(\sqrt{2})^{n-1}}\)

Eccentricity of all the ellipses will be same

Distance of a focus from the centre in \(E_9 = a_9e_9 = \sqrt{a_9^2 - b_9^2} = \frac{\sqrt{5}}{16}\)

Length of latus rectum of \(E_9 = \frac{2b_9^2}{a_9} = \frac{1}{6}\)

\[ \therefore \sum_{n=1}^{N} \text{Area of } R_n = 12 + \frac{12}{2} + \frac{12}{4} + \ldots, \infty = 24 \]

\[ \Rightarrow \sum_{n=1}^{N} (\text{area of } R_n) < 24, \text{ for each positive integer } N \]
3. Let \( M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix} \) and \( \text{adj}M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \) where \( a \) and \( b \) are real numbers. Which of the following options is/are correct?

(1) \( a + b = 3 \)

(2) \( \det(\text{adj}M^2) = 81 \)

(3) \( (\text{adj}M)^{-1} + \text{adj}M^{-1} = -M \)

(4) If \( M = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), then \( \alpha - \beta + \gamma = 3 \)

Ans. (1,3,4)

Sol. \( (\text{adj}M)_{11} = 2 - 3b = -1 \Rightarrow b = 1 \)

Also, \( (\text{adj}M)_{22} = -3a = -6 \Rightarrow a = 2 \)

Now, \( \det M = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2 \)

\[ \Rightarrow \det(\text{adj}M^2) = (\det M^2)^2 = (\det M)^4 = 16 \]

Also \( M^{-1} = \frac{\text{adj}M}{\det M} \)

\[ \Rightarrow \text{adj}M = -2M^{-1} \]

\[ \Rightarrow (\text{adj}M)^{-1} = \frac{-1}{2}M \]

And, \( \text{adj}(M^{-1}) = (M^{-1})^{-1} \det(M^{-1}) = \frac{1}{\det M}M = \frac{-M}{2} \)

Hence, \( (\text{adj}M)^{-1} + \text{adj}(M^{-1}) = -M \)

Further, \( MX = b \)

\[ \Rightarrow X = M^{-1}b = \frac{-\text{adj}M}{2}b = \frac{-1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \]

\[ \Rightarrow (\alpha, \beta, \gamma) = (1, -1, 1) \]
4. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be given by

\[
\begin{align*}
  f(x) &= \begin{cases} 
    x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\
    x^2 - x + 1, & 0 \leq x < 1; \\
    \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\
    (x - 2)\log_e(x - 2) - x + \frac{10}{3}, & x \geq 3 
  \end{cases}
\end{align*}
\]

Then which of the following options is/are correct?

1. \( f' \) has a local maximum at \( x = 1 \)
2. \( f \) is onto
3. \( f \) is increasing on \((-\infty, 0)\)
4. \( f' \) is NOT differentiable at \( x = 1 \)

Ans. (1,2,4)

Sol. \( f(x) = \begin{cases} 
    (x+1)^5 - 2x, & x < 0; \\
    x^2 - x + 1, & 0 \leq x < 1; \\
    \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\
    (x - 2)\log_e(x - 2) - x + \frac{10}{3}, & x \geq 3 
  \end{cases} \)

for \( x < 0 \), \( f(x) \) is continuous

\[
&\lim_{x \to \infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 1
\]

Hence, \((-\infty, 1) \subseteq \text{Range of } f(x) \text{ in } (-\infty, 0)\)

\( f'(x) = 5(x + 1)^4 - 2 \), which changes sign in \((-\infty, 0)\)

\( \Rightarrow \quad f(x) \) is non-monotonic in \((-\infty, 0)\)

For \( x \geq 3 \), \( f(x) \) is again continuous and \( \lim_{x \to \infty} f(x) = \infty \) and \( f(3) = \frac{1}{3} \)

\( \Rightarrow \quad \left[ \frac{1}{3}, \infty \right) \subseteq \text{Range of } f(x) \text{ in } [3, \infty) \)

Hence, range of \( f(x) \) is \( \mathbb{R} \)

\[
  f'(x) = \begin{cases} 
    2x - 1, & 0 \leq x < 1 \\
    2x^2 - 8x + 7, & 1 \leq x < 3 
  \end{cases}
\]

Hence \( f' \) has a local maximum at \( x = 1 \) and \( f' \) is NOT differentiable at \( x = 1 \).
5. Let \( \alpha \) and \( \beta \) be the roots of \( x^2 - x - 1 = 0 \), with \( \alpha > \beta \). For all positive integers \( n \), define

\[
\begin{align*}
    a_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \\
    b_1 &= 1 \quad \text{and} \quad b_n = a_{n-1} + a_{n+1}, \quad n \geq 2.
\end{align*}
\]

Then which of the following options is/are correct?

1. \( a_1 + a_2 + a_3 + \ldots + a_n = a_{n+2} - 1 \) for all \( n \geq 1 \)
2. \( \sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89} \)
3. \( \sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89} \)
4. \( b_n = \alpha^n + \beta^n \) for all \( n \geq 1 \)

Ans. (1,2,4)

Sol. \( \alpha, \beta \) are roots of \( x^2 - x - 1 = 0 \)

\[
\begin{align*}
    a_{r+2} - a_r &= \frac{(\alpha^{r+2} - \beta^{r+2}) - (\alpha^r - \beta^r)}{\alpha - \beta} - \frac{(\alpha^{r+2} - \alpha^r) - (\beta^{r+2} - \beta^r)}{\alpha - \beta} \\
    &= \frac{\alpha^r(\alpha^2 - 1) - \beta^r(\beta^2 - 1)}{\alpha - \beta} \\
    &= \frac{\alpha^r(\alpha - \beta)(\alpha + \beta) - \beta^r(\beta - 1)(\beta + 1)}{\alpha - \beta} \\
    &= \frac{\alpha^r(\alpha - \beta)(\alpha + \beta) - \beta^r(\beta - 1)(\beta + 1)}{\alpha - \beta} \\
    &= \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = a_{r+1}\)
\]

\( \Rightarrow \) \( a_{r+2} - a_{r+1} = a_r \)

\( \Rightarrow \) \( \sum_{r=1}^{n} a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta} = a_{n+2} - (\alpha + \beta) = a_{n+2} - 1 \)

Now \( \sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \left( \frac{\alpha}{10} \right)^n - \sum_{n=1}^{\infty} \left( \frac{\beta}{10} \right)^n \)

\[
\begin{align*}
    &= \frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta} \\
    &= \frac{1 - \frac{\beta}{10}}{\alpha - \beta} - \frac{1 - \frac{\alpha}{10}}{10 - \beta} = \frac{10}{89}
\end{align*}
\]

\( \sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{\sum_{n=1}^{\infty} a_{n-1} + a_{n+1}}{10^n} = \frac{\alpha}{10} + \frac{\beta}{10} = \frac{12}{89} \)

Further, \( b_n = a_{n-1} + a_{n+1} \)

\[
\begin{align*}
    &= \frac{\alpha^{n-1} - \beta^{n-1} + \alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \\
    &= \frac{\alpha^{n-1} - \beta^{n-1} + \alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \quad \text{(as } \alpha \beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n \beta \text{ and } \beta^{n-1} = -\alpha^n) \\
    &= \frac{\alpha^n(\alpha - \beta) + (\alpha - \beta)\beta^n}{\alpha - \beta} = \alpha^n + \beta^n
\end{align*}
\]
6. Let \( \Gamma \) denote a curve \( y = y(x) \) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to \( \Gamma \) at a point \( P \) intersect the y-axis at \( Y_P \). If \( PY_P \) has length 1 for each point \( P \) on \( \Gamma \), then which of the following options is/are correct?

\[
\begin{align*}
(1) \quad y &= \log_e \left( \frac{1+\sqrt{1-x^2}}{x} \right) - \sqrt{1-x^2} \\
(2) \quad xy' - \sqrt{1-x^2} &= 0 \\
(3) \quad y &= -\log_e \left( \frac{1+\sqrt{1-x^2}}{x} \right) + \sqrt{1-x^2} \\
(4) \quad xy' + \sqrt{1-x^2} &= 0
\end{align*}
\]

Ans. (1,4)

Sol.

\[
\begin{align*}
Y - y &= y'(X - x) \\
\text{So, } \quad Y_P &= (0, y - xy') \\
\text{So, } \quad x^2 + (xy')^2 &= 1 \quad \Rightarrow \quad \frac{dy}{dx} = -\sqrt{\frac{1-x^2}{x^2}}
\end{align*}
\]

\( \frac{dy}{dx} \) can not be positive i.e. \( f(x) \) can not be increasing in first quadrant, for \( x \in (0, 1) \)

Hence, \( \int dy = \int -\sqrt{\frac{1-x^2}{x}} \) \( dx \)

\[
\begin{align*}
\Rightarrow y &= \int \frac{\cos^2 \theta}{\sin \theta} \ d\theta \quad \text{; put } x = \sin \theta \\
\Rightarrow y &= -\int \csc \theta \ d\theta + \int \sin \theta \ d\theta \\
\Rightarrow y &= \ln(\csc \theta + \cot \theta) - \cos \theta + C \\
\Rightarrow y &= \ln \left( \frac{1+\sqrt{1-x^2}}{x} \right) - \sqrt{1-x^2} + C \\
\Rightarrow y &= \ln \left( \frac{1+\sqrt{1-x^2}}{x} \right) - \sqrt{1-x^2} \quad \text{(as } y(1) = 0)\]
\]
7. In a non-right-angled triangle $\triangle PQR$, let $p$, $q$, $r$ denote the lengths of the sides opposite to the angles at $P$, $Q$, $R$ respectively. The median from $R$ meets the side $PQ$ at $S$, the perpendicular from $P$ meets the side $QR$ at $E$, and $RS$ and $PE$ intersect at $O$. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the $\triangle PQR$ equals 1, then which of the following options is/are correct?

1. Area of $\triangle SOE = \frac{\sqrt{3}}{12}$
2. Radius of incircle of $\triangle PQR = \frac{\sqrt{3}}{2} (2 - \sqrt{3})$
3. Length of $RS = \frac{\sqrt{7}}{2}$
4. Length of $OE = \frac{1}{6}$

Ans. (2,3,4)

Sol. 
\[
\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}
\]

$\Rightarrow$ P = $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ and $Q = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Since $p > q$ $\Rightarrow$ P > Q

So, if $P = \frac{\pi}{3}$ and $Q = \frac{\pi}{6}$ $\Rightarrow$ $R = \frac{\pi}{2}$ (not possible)

Hence, $P = \frac{2\pi}{3}$ and $Q = R = \frac{\pi}{6}$

\[
r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})(\frac{1}{2})}{\frac{\sqrt{3} + 2}{2}} = \frac{\sqrt{3}}{2} (2 - \sqrt{3})
\]

Now, area of $\triangle SEF = \frac{1}{4}$ area of $\triangle PQR$

$\Rightarrow$ area of $\triangle SOE = \frac{1}{3}$ area of $\triangle SEF = \frac{1}{12}$ area of $\triangle PQR = \frac{1}{12} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$

\[
RS = \frac{1}{2} \sqrt{2(3) + 2(1) - 1} = \frac{\sqrt{7}}{2}
\]

\[
OE = \frac{1}{3} PE = \frac{1}{3} \cdot \frac{1}{2} \sqrt{2(1)^2 + 2(1)^2 - 3} = \frac{1}{6}
\]
8. Let \( L_1 \) and \( L_2 \) denotes the lines
\[
\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in \mathbb{R} \quad \text{and} \quad \\
\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in \mathbb{R}
\]
respectively. If \( L_3 \) is a line which is perpendicular to both \( L_1 \) and \( L_2 \) and cuts both of them, then which of the following options describe(s) \( L_3 \)?

(1) \( \vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R} \)

(2) \( \vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R} \)

(3) \( \vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R} \)

(4) \( \vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R} \)

Ans. \((1,2,4)\)

Sol. Points on \( L_1 \) and \( L_2 \) are respectively \( A(1 - \lambda, 2\lambda, 2\lambda) \) and \( B(2\mu, -\mu, 2\mu) \)

So, \( \overline{AB} = (2\mu + \lambda - 1)\hat{i} + (-\mu - 2\lambda)\hat{j} + (2\mu - 2\lambda)\hat{k} \)

and vector along their shortest distance = \( 2\hat{i} + 2\hat{j} - \hat{k} \).

Hence, \( \frac{2\mu + \lambda - 1}{2} = \frac{-\mu - 2\lambda}{2} = \frac{2\mu - 2\lambda}{-1} \)

\( \Rightarrow \lambda = \frac{1}{9} \quad \text{and} \quad \mu = \frac{2}{9} \)

Hence, \( A = \left( \frac{8}{9}, \frac{2}{9}, \frac{2}{9} \right) \) and \( B = \left( \frac{4}{9}, \frac{-2}{9}, \frac{4}{9} \right) \)

\( \Rightarrow \) Mid point of \( AB = \left( \frac{2}{3}, 0, \frac{1}{3} \right) \)

SECTION-3 : (Maximum Marks: 18)

- This section contains SIX (06) questions. The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate/round-off the value to Two decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  
  * **Full Marks** : +3 if ONLY the correct numerical value is entered.
  * **Zero Marks** : 0 in all other cases.
1. If
\[
I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{ix})(2 - \cos 2x)}
\]
then $27I^2$ equals _____

Ans. (4.00)

Sol. $2I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \left[ \frac{1}{(1 + e^{ix})(2 - \cos 2x)} + \frac{1}{(1 + e^{-ix})(2 - \cos 2x)} \right] dx$ (using King's Rule)

\[
\Rightarrow I = \frac{1}{\pi} \int_{\pi/4}^{\pi/4} \frac{dx}{2 - \cos 2x}
\]

\[
\Rightarrow I = \frac{2}{\pi} \int_0^{\pi/4} \frac{dx}{2 - \cos 2x} = \frac{2}{\pi} \int_0^{\pi/4} \sec^2 x \, dx
\]

\[
= \frac{2}{\sqrt{3}\pi} \left[ \tan^{-1} \left( \sqrt{3} \tan x \right) \right]_0^{\pi/4} = \frac{2}{3\sqrt{3}}
\]

\[
\Rightarrow 27I^2 = 27 \times \frac{4}{27} = 4
\]

2. Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let $\Gamma_A$ and $\Gamma_B$ be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles $\Gamma_A$ and $\Gamma_B$ such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____

Ans. (10.00)

Sol. Distance of point A from given line $= \frac{5}{2}$

\[
\frac{CA}{CB} = \frac{2}{1}
\]

\[
\Rightarrow \frac{AC}{AB} = \frac{2}{1}
\]

\[
\Rightarrow AC = 2 \times 5 = 10
\]
3. Let $AP(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term $a$ and common difference $d > 0$. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then $a + d$ equals ___.

Ans. (157.00)

Sol. We equate the general terms of three respective A.P.'s as $1 + 3a = 2 + 5b = 3 + 7c$

$\Rightarrow$ 3 divides $1 + 2b$ and 5 divides $1 + 2c$

$\Rightarrow$ $1 + 2c = 5, 15, 25$ etc.

So, first such terms are possible when $1 + 2c = 15$ i.e. $c = 7$

Hence, first term $a = 52$

d = lcm (3, 5, 7) = 105

$\Rightarrow$ $a + d = 157$

4. Let $S$ be the sample space of all $3 \times 3$ matrices with entries from the set {0, 1}. Let the events $E_1$ and $E_2$ be given by

$E_1 = \{A \in S : \det A = 0\}$ and

$E_2 = \{A \in S : \text{sum of entries of A is 7}\}$.

If a matrix is chosen at random from $S$, then the conditional probability $P(E_1|E_2)$ equals ____

Ans. (0.50)

Sol. $n(E_2) = 9C_2$ (as exactly two cyphers are there)

Now, $\det A = 0$, when two cyphers are in the same column or same row

$\Rightarrow n(E_1 \cap E_2) = 6 \times 3C_2$

Hence, $P\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{36} = \frac{1}{2} = 0.50$

5. Three lines are given by

$\vec{r} = \lambda \hat{i}, \lambda \in \mathbb{R}$

$\vec{r} = \mu(\hat{i} + \hat{j}), \mu \in \mathbb{R}$ and

$\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k}), \nu \in \mathbb{R}$.

Let the lines cut the plane $x + y + z = 1$ at the points A, B and C respectively. If the area of the triangle ABC is $\Delta$ then the value of $(6\Delta)^2$ equals ___

Ans. (0.75)
Sol. \( A(1, 0, 0), \quad B\left(\frac{1}{2}, \frac{1}{2}, 0\right) \) & \( C\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \)

Hence, \( \overline{AB} = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} \) & \( \overline{AC} = -\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k} \)

So, \( \Delta = \frac{1}{2}|\overline{AB} \times \overline{AC}| = \frac{1}{2} \sqrt{\frac{2}{3} - \frac{1}{4}} \)

\( = \frac{1}{2 \times 2\sqrt{3}} \)

\( \Rightarrow (6\Delta)^2 = \frac{3}{4} = 0.75 \)

6. Let \( \omega \neq 1 \) be a cube root of unity. Then the minimum of the set

\[ \{|a + b\omega + c\omega^2| : a, b, c \text{ distinct non-zero integers}\} \]

equals ____

Ans. (3.00)

Sol. \( |a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)(a + b\omega + c\omega^2) \)

\( = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) \)

\( = a^2 + b^2 + c^2 - ab - bc - ca \)

\( = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] \)

\( \geq \frac{1 + 1 + 4}{2} = 3 \) (when \( a = 1, b = 2, c = 3 \))