FINAL JEE(Advanced) EXAMINATION - 2019

(Held On Monday 27th MAY, 2019)

PAPER-1 | TEST PAPER WITH ANSWER & SOLUTION

PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +3 If ONLY the correct option is chosen.

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered)

Negative Marks : –1 In all other cases

1. Let $M=\begin{bmatrix}\sin^4\theta & -1-\sin^2\theta\\ 1+\cos^2\theta & \cos^4\theta\end{bmatrix}=\alpha I+\beta M^{-1}$,

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real number, and I is the 2×2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

then the value of $\alpha^* + \beta^*$ is

(1) $-\frac{37}{16}$ (2) $-\frac{29}{16}$ (3) $-\frac{31}{16}$ (4) $-\frac{17}{16}$

Ans. **(2)**

Sol. Given $M = \alpha I + \beta M^{-1}$

$\Rightarrow M^2 - \alpha M - \beta I = O$

By putting values of M and M^2, we get

$\alpha(\theta) = 1 - 2\sin^2\theta\cos^2\theta = 1-\frac{\sin^2 2\theta}{2}\geq\frac{1}{2}$

Also, $\beta(\theta) = -(\sin^4\theta\cos^4\theta + (1 + \cos^2\theta)(1 + \sin^2\theta))$

$= -(\sin^4\theta\cos^4\theta + 1 + \cos^2\theta + \sin^2\theta + \sin^2\theta\cos^2\theta)$

$= -(t^2 + t + 2),\ t=\frac{\sin^2 2\theta}{4}\in\left[0,\frac{1}{4}\right]$

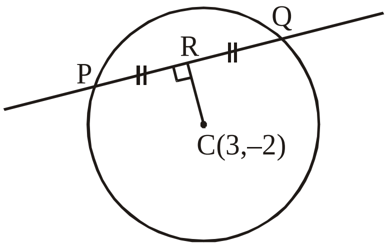
$\Rightarrow \beta(\theta) \geq -\frac{37}{16}$

2. A line y = mx + 1 intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

(1) $6 \le m < 8$ (2) $2 \le m < 4$ (3) $4 \le m < 6$ (4) $-3 \le m < -1$

Ans. (2)

Sol.



$$R \equiv \left(-\frac{3}{5}, \frac{-3m}{5}+1\right)$$

$$\text{So, } m\left(\frac{-\frac{3m}{5}+3}{-\frac{3}{5}-3}\right) = -1$$

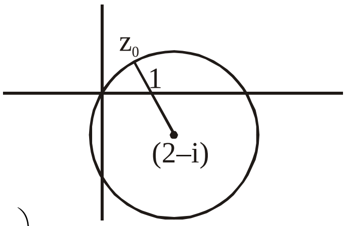
$\Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

3. Let S be the set of all complex numbers z satisfying $|z-2+i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is

(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

Ans. (2)



Sol. $$\arg\left(\frac{4-(z_0-\bar{z}_0)}{(z_0-\bar{z}_0)+zi}\right)$$

$$= \arg\left(\frac{4-2\,\text{Re}\,z_0}{2i\,\text{Im}\,z_0+2i}\right) = \arg\left(\frac{2-\text{Re}\,z_0}{(1+\text{Im}\,z_0)i}\right)$$

$$= \arg\left(-\left(\frac{2-\text{Re}\,z_0}{1+\text{Im}\,z_0}\right)i\right)$$

$= \arg(-ki)$; $k > 0$ (as $\text{Re}z_0 < 2$ & $\text{Im}z_0 > 0$)

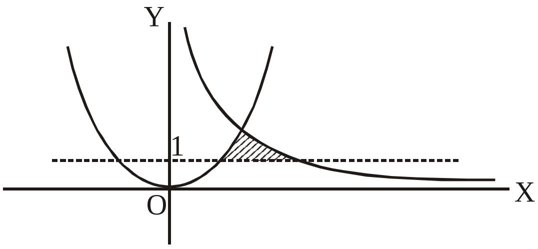
$$= -\frac{\pi}{2}$$

4. The area of the region $\{(x, y) : xy \le 8, 1 \le y \le x^2\}$ is

(1) $8\log_e 2 - \frac{14}{3}$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $16\log_e 2 - 6$ (4) $8\log_e 2 - \frac{7}{3}$

Ans. (2)

Sol.



For intersection, $\frac{8}{y} = \sqrt{y} \Rightarrow y = 4$

Hence, required area = $\int_1^4 \left(\frac{8}{y} - \sqrt{y}\right) dy$

$$= \left[8\ell ny - \frac{2}{3}y^{3/2}\right]_1^4 = 16\ell n2 - \frac{14}{3}$$

Remark : The question should contain the phrase "area of the bounded region in the first quadrant". Because, in the 2nd quadrant, the region above the line y = 1 and below $y = x^2$, satisfies the region, which is unbounded.

SECTION-2 : (Maximum Marks: 32)

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s)
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: +3	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: +2	If three or more options are correct but ONLY two options are chosen and both of which are correct.
Partial Marks	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: 0	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	: –1	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2 marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 marks;
choosing ONLY (B) will get +1 marks;
choosing ONLY (D) will get +1 marks;
choosing no option (i.e. the question is unanswered) will get 0 marks, and
choosing any other combination of options will get –1 mark.

1. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls, Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

(1) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(2) Probability that the chosen ball is green equals $\frac{39}{80}$

(3) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(4) Probability that the selected bag is B_3, given that the chosen balls is green, equals $\frac{5}{13}$

Ans. (2,3)

Sol.

Ball	Balls composition	$P(B_i)$
B_1	$5R+5G$	$\frac{3}{10}$
B_2	$3R+5G$	$\frac{3}{10}$
B_3	$5R+3G$	$\frac{4}{10}$

(1) $$P(B_3 \cap G) = P\left(\frac{G_1}{B_3}\right)P(B_3)$$

$$= \frac{3}{8} \times \frac{4}{10} = \frac{3}{20}$$

(2) $$P(G) = P\left(\frac{G_1}{B_1}\right)P(B_1) + P\left(\frac{G}{B_2}\right)P(B_2) + P\left(\frac{G}{B_3}\right)P(B_3)$$

$$= \frac{3}{20} + \frac{3}{16} + \frac{3}{20} = \frac{39}{80}$$

(3) $$P\left(\frac{G}{B_3}\right) = \frac{3}{8}$$

(4) $$P\left(\frac{B_3}{G}\right) = \frac{P(G \cap B_3)}{P(G)} = \frac{3/20}{39/80} = \frac{4}{13}$$

2. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct ?

(1) The eccentricities of E_{18} and E_{19} are NOT equal

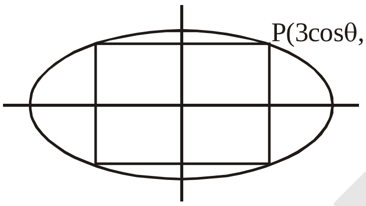
(2) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) $\sum_{n=1}^{N} (\text{area of } R_n) < 24$, for each positive integer N

Ans. (3,4)

Sol.



Area of R_1 = $3\sin 2\theta$; for this to be maximum

$\Rightarrow \theta = \frac{\pi}{4} \Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

Hence for subsequent areas of rectangles R_n to be maximum the coordinates will be in GP with common ratio $r = \frac{1}{\sqrt{2}} \Rightarrow a_n = \frac{3}{(\sqrt{2})^{n-1}}$; $b_n = \frac{3}{(\sqrt{2})^{n-1}}$

Eccentricity of all the ellipses will be same

Distance of a focus from the centre in $E_9 = a_9 e_9 = \sqrt{a_9^2 - b_9^2} = \frac{\sqrt{5}}{16}$

Length of latus rectum of $E_9 = \frac{2b_9^2}{a_9} = \frac{1}{6}$

$\because \sum_{n=1}^{\infty} \text{Area of } R_n = 12 + \frac{12}{2} + \frac{12}{4} + ...\infty = 24$

$\Rightarrow \sum_{n=1}^{N} (\text{area of } R_n) < 24$, for each positive integer N

3. Let $M=\begin{bmatrix}0 & 1 & a\\ 1 & 2 & 3\\ 3 & b & 1\end{bmatrix}$ and $\text{adj}M=\begin{bmatrix}-1 & 1 & -1\\ 8 & -6 & 2\\ -5 & 3 & -1\end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ?

(1) $a + b = 3$

(2) $\det(\text{adj}M^2) = 81$

(3) $(\text{adj}M)^{-1} + \text{adj}M^{-1} = -M$

(4) If $M\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

Ans. (1,3,4)

Sol. $(\text{adj}M)_{11} = 2 - 3b = -1 \Rightarrow b = 1$

Also, $(\text{adj}M)_{22} = -3a = -6 \Rightarrow a = 2$

Now, $\det M=\begin{vmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{vmatrix}=-2$

$\Rightarrow \det(\text{adj}M^2) = (\det M^2)^2$

$= (\det M)^4 = 16$

Also $M^{-1}=\dfrac{\text{adj}M}{\det M}$

$\Rightarrow \text{adj}M = -2M^{-1}$

$\Rightarrow (\text{adj}M)^{-1} = \dfrac{-1}{2}M$

And, $\text{adj}(M^{-1}) = (M^{-1})^{-1}\det(M^{-1})$

$=\dfrac{1}{\det M}M=\dfrac{-M}{2}$

Hence, $(\text{adj}M)^{-1} + \text{adj}(M^{-1}) = -M$

Further, $MX = b$

$\Rightarrow X = M^{-1}b = \dfrac{-\text{adj}M}{2}b$

$=\dfrac{-1}{2}\begin{bmatrix}-1 & 1 & -1\\ 8 & -6 & 2\\ -5 & 3 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

$=\dfrac{-1}{2}\begin{bmatrix}-2\\ 2\\ -2\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$

$\Rightarrow (\alpha, \beta, \gamma) = (1, -1, 1)$

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x)=\begin{cases} x^5+5x^4+10x^3+10x^2+3x+1, & x<0; \\ x^2-x+1, & 0\le x<1; \\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3}, & 1\le x<3; \\ (x-2)\log_e(x-2)-x+\frac{10}{3}, & x\ge 3 \end{cases}$$

Then which of the following options is/are correct ?

(1) f' has a local maximum at x = 1

(2) f is onto

(3) f is increasing on $(-\infty, 0)$

(4) f' is NOT differentiable at x = 1

Ans. (1,2,4)

Sol. $$f(x)=\begin{cases} (x+1)^5-2x, & x<0; \\ x^2-x+1, & 0\le x<1; \\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3}, & 1\le x<3; \\ (x-2)\log_e(x-2)-x+\frac{10}{3}, & x\ge 3 \end{cases}$$

for x < 0, $f(x)$ is continuous

& $\lim\limits_{x\to-\infty} f(x)=-\infty$ and $\lim\limits_{x\to0^-} f(x)=1$

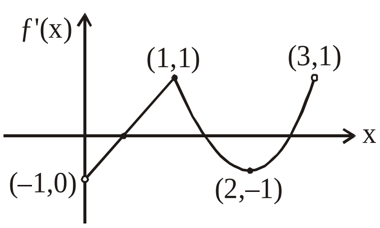
Hence, $(-\infty, 1) \subset$ Range of $f(x)$ in $(-\infty, 0)$

$f'(x) = 5(x+1)^4 - 2$, which changes sign in $(-\infty, 0)$

$\Rightarrow$ $f(x)$ is non-monotonic in $(-\infty, 0)$

For $x \ge 3$, $f(x)$ is again continuous and $\lim\limits_{x\to\infty} f(x)=\infty$ and $f(3) = \frac{1}{3}$

$\Rightarrow$ $\left[\frac{1}{3},\infty\right) \subset$ Range of $f(x)$ in $[3, \infty)$



Hence, range of $f(x)$ is $\mathbb{R}$

$$f'(x)=\begin{cases} 2x-1, & 0\le x<1 \\ 2x^2-8x+7, & 1\le x<3 \end{cases}$$

Hence f' has a local maximum at x = 1 and f' is NOT differentiable at x = 1.

5. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\quad n \geq 1$$

$$b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1},\ n \geq 2.$$

Then which of the following options is/are correct ?

(1) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(2) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(4) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

Ans. (1,2,4)

Sol. α, β are roots of $x^2 - x - 1$

$$a_{r+2} - a_r = \frac{\left(\alpha^{r+2} - \beta^{r+2}\right) - \left(\alpha^r - \beta^r\right)}{\alpha - \beta} = \frac{\left(\alpha^{r+2} - \alpha^r\right) - \left(\beta^{r+2} - \beta^r\right)}{\alpha - \beta}$$

$$= \frac{\alpha^r\left(\alpha^2 - 1\right) - \beta^r\left(\beta^2 - 1\right)}{\alpha - \beta} = \frac{\alpha^r\alpha - \beta^r\beta}{\alpha - \beta} = \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = a_{r+1}$$

$$\Rightarrow a_{r+2} - a_{r+1} = a_r$$

$$\Rightarrow \sum_{r=1}^{n} a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta} = a_{n+2} - (\alpha + \beta) = a_{n+2} - 1$$

Now $$\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{\sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n - \sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n}{\alpha - \beta}$$

$$= \frac{\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}}}{\alpha - \beta} = \frac{\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta}}{(\alpha - \beta)} = \frac{10}{(10 - \alpha)(10 - \beta)} = \frac{10}{89}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \frac{a_{n-1} + a_{n+1}}{10^n} = \frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} = \frac{12}{89}$$

Further, $b_n = a_{n-1} + a_{n+1}$

$$= \frac{\left(\alpha^{n-1} - \beta^{n-1}\right) + \left(\alpha^{n+1} - \beta^{n+1}\right)}{\alpha - \beta}$$

(as $\alpha\beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n\beta$ & $\beta^{n-1} = -\alpha\beta^n$)

$$= \frac{\alpha^n(\alpha - \beta) + (\alpha - \beta)\beta^n}{\alpha - \beta} = \alpha^n + \beta^n$$

6. Let Γ denote a curve y = y(x) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ?

(1) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

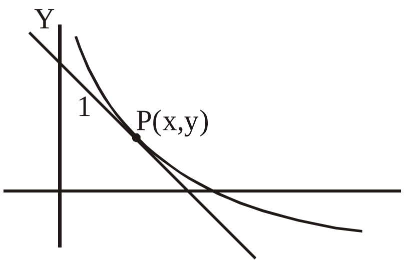
(2) $xy' - \sqrt{1-x^2} = 0$

(3) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(4) $xy' + \sqrt{1-x^2} = 0$

Ans. (1,4)

Sol.



$Y - y = y'(X - x)$

So, $Y_P = (0,\ y - xy')$

So, $x^2 + (xy')^2 = 1 \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-x^2}{x^2}}$

[$\frac{dy}{dx}$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x \in (0, 1)$]

Hence, $\int dy = -\int \frac{\sqrt{1-x^2}}{x}\,dx$

$\Rightarrow y = -\int \frac{\cos^2\theta\, d\theta}{\sin\theta}$; put x = sinθ

$\Rightarrow y = -\int \text{cosec}\theta\, d\theta + \int \sin\theta\, d\theta$

$\Rightarrow y = \ln(\text{cosec}\theta + \cot\theta) - \cos\theta + C$

$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + C$

$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$ (as y(1) = 0)

7. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, $q=1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

(1) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

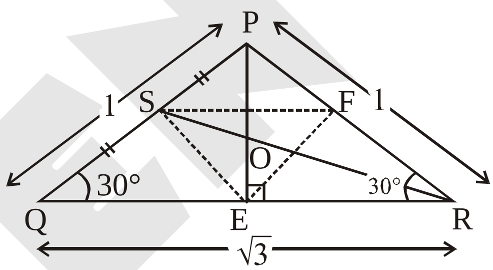
(2) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2-\sqrt{3})$

(3) Length of $RS = \frac{\sqrt{7}}{2}$

(4) Length of $OE = \frac{1}{6}$

Ans. (2,3,4)

Sol. $\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$



$\Rightarrow \quad P = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ and $Q = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Since $p > q \Rightarrow P > Q$

So, if $P = \frac{\pi}{3}$ and $Q = \frac{\pi}{6} \quad \Rightarrow \quad R = \frac{\pi}{2}$ (not possible)

Hence, $P = \frac{2\pi}{3}$ and $Q = R = \frac{\pi}{6}$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}(2-\sqrt{3})$$

Now, area of $\Delta SEF = \frac{1}{4}$ area of ΔPQR

$\Rightarrow$ area of $\Delta SOE = \frac{1}{3}$ area of $\Delta SEF = \frac{1}{12}$ area of $\Delta PQR = \frac{1}{12}\cdot\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$

$$RS = \frac{1}{2}\sqrt{2(3)+2(1)-1} = \frac{\sqrt{7}}{2}$$

$$OE = \frac{1}{3}PE = \frac{1}{3}\cdot\frac{1}{2}\sqrt{2(1)^2+2(1)^2-3} = \frac{1}{6}$$

8. Let L_1 and L_2 denotes the lines

$$\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in \mathbb{R} \text{ and}$$

$$\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in \mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

(1) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$

(2) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$

(3) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$

(4) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in \mathbb{R}$

Ans. (1,2,4)

Sol. Points on L_1 and L_2 are respectively $A(1 - \lambda, 2\lambda, 2\lambda)$ and $B(2\mu, -\mu, 2\mu)$

So, $\overrightarrow{AB} = (2\mu + \lambda - 1)\hat{i} + (-\mu - 2\lambda)\hat{j} + (2\mu - 2\lambda)\hat{k}$

and vector along their shortest distance $= 2\hat{i} + 2\hat{j} - \hat{k}$.

Hence, $\frac{2\mu + \lambda - 1}{2} = \frac{-\mu - 2\lambda}{2} = \frac{2\mu - 2\lambda}{-1}$

$\Rightarrow \quad \lambda = \frac{1}{9} \ \& \ \mu = \frac{2}{9}$

Hence, $A \equiv \left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right)$ and $B \equiv \left(\frac{4}{9}, -\frac{2}{9}, \frac{4}{9}\right)$

$\Rightarrow \quad$ Mid point of $AB \equiv \left(\frac{2}{3}, 0, \frac{1}{3}\right)$

SECTION-3 : (Maximum Marks: 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **Two** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +3	If ONLY the correct numerical value is entered.
Zero Marks	: 0	In all other cases.

1. If

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$$

then $27I^2$ equals _____

Ans. (4.00)

Sol. $$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[\frac{1}{(1+e^{\sin x})(2-\cos 2x)} + \frac{1}{(1+e^{-\sin x})(2-\cos 2x)}\right]dx \text{ (using King's Rule)}$$

$$\Rightarrow \quad I = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x}$$

$$\Rightarrow \quad I = \frac{2}{\pi}\int_{0}^{\pi/4}\frac{dx}{2-\cos 2x} = \frac{2}{\pi}\int_{0}^{\pi/4}\frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

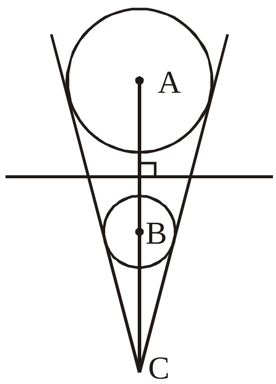
$$= \frac{2}{\sqrt{3}\pi}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/4} = \frac{2}{3\sqrt{3}}$$

$$\Rightarrow \quad 27I^2 = 27\times\frac{4}{27} = 4$$

2. Let the point B be the reflection of the point A(2, 3) with respect to the line 8x – 6y – 23 = 0. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____

Ans. (10.00)

Sol. Distance of point A from given line $= \frac{5}{2}$



$$\frac{CA}{CB} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{2}{1}$$

$$\Rightarrow \quad AC = 2\times 5 = 10$$

3. Let AP (a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then a + d equals ___

Ans. (157.00)

Sol. We equate the general terms of three respective

A.P.'s as $1 + 3a = 2 + 5b = 3 + 7c$

$\Rightarrow$ 3 divides $1 + 2b$ and 5 divides $1 + 2c$

$\Rightarrow$ $1 + 2c = 5, 15, 25$ etc.

So, first such terms are possible when $1 + 2c = 15$ i.e. $c = 7$

Hence, first term = a = 52

d = lcm (3, 5, 7) = 105

$\Rightarrow$ a + d = 157

4. Let S be the sample space of all 3 × 3 matrices with entries from the set {0, 1}. Let the events E_1 and E_2 be given by

$$E_1 = \{A \in S : \det A = 0\} \text{ and}$$

$$E_2 = \{A \in S : \text{sum of entries of A is 7}\}.$$

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ____

Ans. (0.50)

Sol. $n(E_2) = {}^9C_2$ (as exactly two cyphers are there)

Now, det A = 0, when two cyphers are in the same column or same row

$\Rightarrow$ $n(E_1 \cap E_2) = 6 \times {}^3C_2$.

Hence, $P\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{36} = \frac{1}{2} = 0.50$

5. Three lines are given by

$$\vec{r} = \lambda\hat{i}, \lambda \in \mathbb{R}$$

$$\vec{r} = \mu(\hat{i} + \hat{j}), \mu \in \mathbb{R} \text{ and}$$

$$\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k}), \nu \in \mathbb{R}.$$

Let the lines cut the plane x + y + z = 1 at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals ___

Ans. (0.75)

Sol. A(1, 0, 0), $B\left(\frac{1}{2},\frac{1}{2},0\right)$ & $C\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Hence, $\overrightarrow{AB}=-\frac{1}{2}\hat{i}+\frac{1}{2}\hat{j}$ & $\overrightarrow{AC}=-\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{3}\hat{k}$

So, $\Delta=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{\frac{1}{2}\times\frac{2}{3}-\frac{1}{4}}$

$=\frac{1}{2\times2\sqrt{3}}$

$\Rightarrow\quad(6\Delta)^2=\frac{3}{4}=0.75$

6. Let $\omega\neq1$ be a cube root of unity. Then the minimum of the set

$\{|a+b\omega+c\omega^2|^2$: a, b, c distinct non-zero integers$\}$

equals ____

Ans. (3.00)

Sol. $|a+b\omega+c\omega^2|^2=(a+b\omega+c\omega^2)\left(\overline{a+b\omega+c\omega^2}\right)$

$=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

$=a^2+b^2+c^2-ab-bc-ca$

$=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

$\geq\frac{1+1+4}{2}=3$ (when a = 1, b = 2, c = 3)