PART-3 : MATHEMATICS

SECTION-1 : (Maximum Marks : 12)

This section contains FOUR (04) questions.

Each question has FOUR options. ONLY ONE of these four options is the correct answer.

For each question, choose the option corresponding to the correct answer.

Answer to each question will be evaluated according to the following marking scheme :

<table>
<thead>
<tr>
<th>Marks</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>+3</td>
<td>If ONLY the correct option is chosen.</td>
</tr>
<tr>
<td>0</td>
<td>If none of the options is chosen (i.e. the question is unanswered)</td>
</tr>
<tr>
<td>-1</td>
<td>In all other cases</td>
</tr>
</tbody>
</table>

1. Let \( M = \begin{bmatrix} \sin^4 \theta & -1 - \sin^2 \theta \\ 1 + \cos^2 \theta & \cos^4 \theta \end{bmatrix} = \alpha I + \beta M^{-1} \),

where \( \alpha = \alpha(\theta) \) and \( \beta = \beta(\theta) \) are real number, and \( I \) is the 2 × 2 identity matrix. If

\( \alpha^* \) is the minimum of the set \( \{ \alpha(\theta) : \theta \in [0, 2\pi) \} \) and

\( \beta^* \) is the minimum of the set \( \{ \beta(\theta) : \theta \in [0, 2\pi) \} \),

then the value of \( \alpha^* + \beta^* \) is

\[
(1) \frac{37}{16} \quad (2) \frac{29}{16} \quad (3) \frac{31}{16} \quad (4) \frac{17}{16}
\]

Ans. (2)

2. A line \( y = mx + 1 \) intersects the circle \( (x - 3)^2 + (y + 2)^2 = 25 \) at the points P and Q. If the midpoint of the line segment PQ has x-coordinate \( \frac{3}{5} \), then which one of the following options is correct ?

\( (1) \ 6 \leq m < 8 \quad (2) \ 2 \leq m < 4 \quad (3) \ 4 \leq m < 6 \quad (4) \ -3 \leq m < -1 \)

Ans. (2)

3. Let \( S \) be the set of all complex numbers \( z \) satisfying \( |z - 2 + i| \geq \sqrt{5} \). If the complex number \( z_0 \) is such that

\( \frac{1}{|z_0 - 1|} \) is the maximum of the set

\( \left\{ \frac{1}{|z - 1|} : z \in S \right\} \), then the principal argument of \( \frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i} \) is

\[
(1) \frac{\pi}{4} \quad (2) \frac{-\pi}{2} \quad (3) \frac{3\pi}{4} \quad (4) \frac{\pi}{2}
\]

Ans. (2)
4. The area of the region \( \{(x, y) : xy \leq 8, 1 \leq y \leq x^2\} \) is

(1) \( 8 \log_e 2 - \frac{14}{3} \) 
(2) \( 16 \log_e 2 - \frac{14}{3} \) 
(3) \( 16 \log_e 2 - 6 \) 
(4) \( 8 \log_e 2 - \frac{7}{3} \)

Ans. (2)

SECTION-2 : (Maximum Marks: 32)

This section contains EIGHT (08) questions.

Each question has FOUR options. ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).

For each question, choose the option(s) corresponding to (all) the correct answer(s).

Answer to each question will be evaluated according to the following marking scheme:

- **Full Marks**: +4 If only (all) the correct option(s) is (are) chosen.
- **Partial Marks**: +3 If all the four options are correct but ONLY three options are chosen.
- **Partial Marks**: +2 If three or more options are correct but ONLY two options are chosen and both of which are correct.
- **Partial Marks**: +1 If two or more options are correct but ONLY one option is chosen and it is a correct option.
- **Zero Marks**: 0 If none of the options is chosen (i.e. the question is unanswered).
- **Negative Marks**: –1 In all other cases.

For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
- choosing ONLY (A), (B) and (D) will get +4 marks;
- choosing ONLY (A) and (B) will get +2 marks;
- choosing ONLY (A) and (D) will get +2 marks;
- choosing ONLY (B) and (D) will get +2 marks;
- choosing ONLY (A) will get +1 marks;
- choosing ONLY (B) will get +1 marks;
- choosing ONLY (D) will get +1 marks;
- choosing no option (i.e. the question is unanswered) will get 0 marks, and
- choosing any other combination of options will get –1 mark.

1. There are three bags \( B_1, B_2 \) and \( B_3 \). The bag \( B_1 \) contains 5 red and 5 green balls, \( B_2 \) contains 3 red and 5 green balls, and \( B_3 \) contains 5 red and 3 green balls. Bags \( B_1, B_2 \) and \( B_3 \) have probabilities \( \frac{3}{10}, \frac{3}{10} \) and \( \frac{4}{10} \) respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct?

(1) Probability that the selected bag is \( B_3 \) and the chosen ball is green equals \( \frac{3}{10} \)

(2) Probability that the chosen ball is green equals \( \frac{39}{80} \)

(3) Probability that the chosen ball is green, given that the selected bag is \( B_3 \), equals \( \frac{3}{8} \)

(4) Probability that the selected bag is \( B_3 \), given that the chosen balls is green, equals \( \frac{5}{13} \)

Ans. (2,3)
2. Define the collections \{E_1, E_2, E_3, \ldots \} of ellipses and \{R_1, R_2, R_3, \ldots \} of rectangles as follows:

\[
E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1 ;
\]

\[
R_1 : \text{rectangle of largest area, with sides parallel to the axes, inscribed in } E_1 ;
\]

\[
E_n : \text{ellipse } \frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1 \text{ of largest area inscribed in } R_{n-1}, n > 1 ;
\]

\[
R_n : \text{rectangle of largest area, with sides parallel to the axes, inscribed in } E_n, n > 1.
\]

Then which of the following options is/are correct?

(1) The eccentricities of \(E_{18}\) and \(E_{19}\) are NOT equal

(2) The distance of a focus from the centre in \(E_9\) is \(\frac{\sqrt{5}}{32}\)

(3) The length of latus rectum of \(E_9\) is \(\frac{1}{6}\)

(4) \(\sum_{n=1}^{N} \text{area of } R_n < 24\), for each positive integer N

Ans. (3,4)

3. Let \(M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}\) and \(\text{adj}M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\) where a and b are real numbers. Which of the following options is/are correct?

(1) \(a + b = 3\)

(2) \(\det(\text{adj}M^2) = 81\)

(3) \((\text{adj}M)^{-1} + \text{adj}M^{-1} = -M\)

(4) If \(\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\), then \(\alpha - \beta + \gamma = 3\)

Ans. (1,3,4)

4. Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be given by

\[
f(x) = \begin{cases} 
  x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\
  x^2 - x + 1, & 0 \leq x < 1; \\
  \frac{2}{3} x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\
  (x - 2) \log_e (x - 2) - x + \frac{10}{3}, & x \geq 3
\end{cases}
\]

Then which of the following options is/are correct?

(1) \(f'\) has a local maximum at \(x = 1\)

(2) \(f\) is onto

(3) \(f\) is increasing on \((-\infty, 0)\)

(4) \(f'\) is NOT differentiable at \(x = 1\)

Ans. (1,2,4)
5. Let $\alpha$ and $\beta$ be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers $n$, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

and

$$b_1 = 1 \quad \text{and} \quad b_n = a_{n-1} + a_{n+1}, \quad n \geq 2.$$

Then which of the following options is/are correct?

(1) $a_1 + a_2 + a_3 + \ldots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(2) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(4) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

Ans. (1,2,4)

6. Let $\Gamma$ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to $\Gamma$ at a point $P$ intersect the y-axis at $Y_p$. If $PY_p$ has length 1 for each point $P$ on $\Gamma$, then which of the following is/are correct?

(1) $y = \log_e\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) - \sqrt{1 - x^2}$

(2) $xy - \sqrt{1 - x^2} = 0$

(3) $y = -\log_e\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) + \sqrt{1 - x^2}$

(4) $xy + \sqrt{1 - x^2} = 0$

Ans. (1,4)

7. In a non-right-angle triangle $\triangle PQR$, let $p$, $q$, $r$ denote the lengths of the sides opposite to the angles at $P$, $Q$, $R$ respectively. The median from $R$ meets the side $PQ$ at $S$, the perpendicular from $P$ meets the side $QR$ at $E$, and $RS$ and $PE$ intersect at $O$. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the $\triangle PQR$ equals 1, then which of the following options is/are correct?

(1) Area of $\triangle SOE = \frac{\sqrt{3}}{12}$

(2) Radius of incircle of $\triangle PQR = \frac{\sqrt{3}}{2} (2 - \sqrt{3})$

(3) Length of $RS = \frac{\sqrt{7}}{2}$

(4) Length of $OE = \frac{1}{6}$

Ans. (2,3,4)
8. Let \( L_1 \) and \( L_2 \) denote the lines
\[
\mathbf{r} = \mathbf{i} + \lambda(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}), \lambda \in \mathbb{R} \quad \text{and} \\
\mathbf{r} = \mu(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}), \mu \in \mathbb{R}
\]
respectively. If \( L_3 \) is a line which is perpendicular to both \( L_1 \) and \( L_2 \) and cuts both of them, then which of the following options describe(s) \( L_3 \) ?

(1) \( \mathbf{r} = \frac{1}{3}(2\mathbf{i} + \mathbf{k}) + t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}), t \in \mathbb{R} \)

(2) \( \mathbf{r} = \frac{2}{9}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}), t \in \mathbb{R} \)

(3) \( \mathbf{r} = t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}), t \in \mathbb{R} \)

(4) \( \mathbf{r} = \frac{2}{9}(4\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}), t \in \mathbb{R} \)

Ans. (1,2,4)

SECTION-3 : (Maximum Marks: 18)

- This section contains SIX (06) questions. The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate/round-off the value to Two decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  - Full Marks : +3 If ONLY the correct numerical value is entered.
  - Zero Marks : 0 In all other cases.

1. If
\[
I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{\sin x})(2 - \cos 2x)}
\]
then \( 27I^2 \) equals _____

Ans. (4.00)

2. Let the point \( B \) be the reflection of the point \( A(2, 3) \) with respect to the line \( 8x - 6y - 23 = 0 \). Let \( \Gamma_A \) and \( \Gamma_B \) be circle of radii 2 and 1 with centres \( A \) and \( B \) respectively. Let \( T \) be a common tangent to the circle \( \Gamma_A \) and \( \Gamma_B \) such that both the circle are on the same side of \( T \). If \( C \) is the point of intersection of \( T \) and the line passing through \( A \) and \( B \), then the length of the line segment \( AC \) is _____

Ans. (10.00)
3. Let \( AP(a; d) \) denote the set of all the terms of an infinite arithmetic progression with first term \( a \) and common difference \( d > 0 \). If \( AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d) \) then \( a + d \) equals \( \boxed{157.00} \)

Ans. \( 157.00 \)

4. Let \( S \) be the sample space of all \( 3 \times 3 \) matrices with entries from the set \( \{0, 1\} \). Let the events \( E_1 \) and \( E_2 \) be given by

\[
E_1 = \{ A \in S : \det A = 0 \} \quad \text{and} \\
E_2 = \{ A \in S : \text{sum of entries of } A \text{ is } 7 \}.
\]

If a matrix is chosen at random from \( S \), then the conditional probability \( P(E_1|E_2) \) equals \( \boxed{0.500} \)

Ans. \( 0.500 \)

5. Three lines are given by

\[
\hat{r} = \lambda \hat{i}, \lambda \in \mathbb{R} \\
\hat{r} = \mu (\hat{i} + \hat{j}), \mu \in \mathbb{R} \quad \text{and} \\
\hat{r} = \nu (\hat{i} + \hat{j} + \hat{k}), \nu \in \mathbb{R}.
\]

Let the lines cut the plane \( x + y + z = 1 \) at the points \( A, B \) and \( C \) respectively. If the area of the triangle \( ABC \) is \( \Delta \) then the value of \( (6\Delta)^2 \) equals \( \boxed{0.75} \)

Ans. \( 0.75 \)

6. Let \( \omega \neq 1 \) be a cube root of unity. Then the minimum of the set

\[
\{|a + b\omega + c\omega^2| : a, b, c \text{ distinct non-zero integers}\}
\]

equals \( \boxed{3.00} \)

Ans. \( 3.00 \)